JOURNAL OF APPROXIMATION THEORY **91**, 1–47 (1997)
ARTICLE NO. AT963087

Forward and Converse Theorems of Polynomial Approximation for Exponential Weights on $[-1, 1]$, I

D. S. Lubinsky

Department of Mathematics, Witwatersrand University, WITS 2050, South Africa
E-mail: 036DSL@COSMOS.WITS.AC.ZA

Communicated by M. v. Golitschek

Received September 26, 1995; accepted in revised form October 15, 1996

We consider exponential weights of the form $w := e^{-Q}$ on $(-1, 1)$ where $Q(x)$ is even and grows faster than $(1-x^2)^{-\delta}$ near ± 1, some $\delta > 0$. For example, we can take

$$Q(x) := \exp_k((1-x^2)^{-\alpha}), \qquad k \geqslant 0, \alpha > 0,$$

where $\exp_k$ denotes the kth iterated exponential and $\exp_0(x) = x$. We prove Jackson theorems in weighted L_p spaces with norm $\|fw\|_{L_p(-1,1)}$ for all $0 < p \leqslant \infty$. In part II of this paper, we shall prove matching converse theorems.

1. STATEMENT OF RESULTS

There is a well developed theory of weighted polynomial approximation for weights $w: (-1, 1) \to (0, \infty)$ that behave like Jacobi weights near ± 1 [6]. However, for weights that decay rapidly near ± 1, this theory does not apply. In this paper, we prove Jackson theorems for even weights

$$w := e^{-Q} \tag{1.1}$$

where $Q: (-1, 1) \to \mathbb{R}$ is even and grows at least as fast as $(1-x^2)^{-\delta}$, some $\delta > 0$, near ± 1. That is, we estimate

$$E_n[f]_{w,p} := \inf_{P \in \mathscr{P}_n} \|(f-P)w\|_{L_p(-1,1)}, \tag{1.2}$$

$0 < p \leqslant \infty$, where $\mathscr{P}_n$ denote the polynomials of degree at most n.

0021-9045/97 $25.00

In some senses, these weights are closer to weights on $\mathbb{R}$, such as $\exp(-\exp(x^2))$, the so-called Erdös weights on $\mathbb{R}$, than to the classical Jacobi weights on $(-1, 1)$. This is borne out by the behaviour of the orthogonal polynomials for these weights. For further orientation on this topic, see [6, 8, 10, 11, 15, 16, 18].

Our methods are similar to those in [5], where Jackson theorems were proved for Freud weights, and to the follow up paper [2], where Erdös weights were treated. The approach involves approximating f by a spline (or piecewise polynomial), representing the piecewise polynomial in terms of certain characteristic functions, and then approximating the characteristic functions (in a suitable sense) by polynomials. This method has the advantage of involving only hypotheses on Q', in contrast with the more complicated approach via orthogonal polynomials and de la Vallee Poussin sums, which typically involves hypotheses on Q'' [6, 15].

To state our result, we need to define our class of weights, as well as various quantities. First, we say that a function $f: (a, b) \to (0, \infty)$ is *quasi-increasing* if $\exists\, C > 0$ such that

$$a < x < y < b \Rightarrow f(x) \leqslant Cf(y).$$

DEFINITION 1.1. Let $w := e^{-Q}$, where

(a) $Q: (-1, 1) \to \mathbb{R}$ is even, is continuous, and has limit ∞ at 1, and Q' is positive in $(0, 1)$.

(b) $xQ'(x)$ is strictly increasing in $(0, 1)$ with right limit 0 at 0.

(c) The function

$$T(x) := \frac{Q'(x)}{Q(x)} \tag{1.3}$$

is quasi-increasing in $(C, 1)$ for some $0 < C < 1$.

(d) $\exists\, C_1, C_2, C_3 > 0$ such that

$$\frac{Q'(y)}{Q'(x)} \leqslant C_1 \left(\frac{Q(y)}{Q(x)}\right)^{C_2}, \qquad y \geqslant x \geqslant C_3. \tag{1.4}$$

(e) For some $\delta > 0$, $0 < C < 1$, $(1 - x^2)^{1+\delta} Q'(x)$ is increasing in $(C, 1)$. Then we write $w \in \mathscr{E}$.

The archetypal example of $w \in \mathscr{E}$ is

$$w(x) := w_{k,\alpha}(x) := \exp(-\exp_k([1 - x^2]^{-\alpha})), \qquad k \geqslant 0, \quad \alpha > 0, \tag{1.5}$$

where $\exp_k = \exp(\exp(\cdots))$ denotes the kth iterated exponential and $\exp_0(x) = x$. For this weight, we see that

$$T(x) = 2\alpha x(1-x^2)^{-1-\alpha} \prod_{j=1}^{k-1} \exp_j([1-x^2]^{-\alpha}), \qquad x > 0.$$

It is not too difficult to see that we can choose $C_2 > 1$ in (1.4) arbitrarily close to 1 in this case, if $k \geqslant 1$. More generally, the function $T(x)$ measures the regularity of growth of $Q(x)$.

We need the condition that $xQ'(x)$ is strictly increasing to guarantee the existence of the *Mhaskar–Rahmanov–Saff number* a_u, the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1-t^2}}, \qquad u > 0. \tag{1.6}$$

If we used something other than a_u (such as Freud's quantity q_u, the root of $u = q_u Q'(q_u)$, or $Q^{[-1]}(u)$, where $Q^{[-1]}$ is the inverse of Q on $(0, 1)$), we could require less of $xQ'(x)$, namely, that it be quasi-increasing for x close to 1. However, this would complicate formulations and it is unlikely that one can still describe the improvement in the degree of aproximation near $\pm a_n$. For those to whom a_u is new, its significance lies partly in the identity [12–14]

$$\|Pw\|_{L_\infty(-1,1)} = \|Pw\|_{L_\infty(-a_n, a_n)}, \qquad P \in \mathscr{P}_n \tag{1.7}$$

and the fact that a_n is the "smallest" such number.

Note that (1.4) on its own forces $Q'(x)$ to grow faster than $(1-x^2)^{-1-\delta}$ near ± 1, for some $\delta > 0$, so there is some overlap between it and condition (e) of Definition 1.1. We need (e) only in Section 5, in constructing polynomial approximations to w^{-1}. We could replace (e) by the implicit assumption that there exist polynomials P_n of degree $O(n)$ such that

$$C_1 \leqslant P_n(x)\, w(x) \leqslant C_2, x \in [-a_n, a_n].$$

In all probability, (a) to (d) of Definition 1.1 already guarantee the existence of such polynomials, and possibly the methods of Totik [19] can be used to verify this.

Our modulus of continuity involves two parts, a "main part" and a "tail." The main part involves rth symmetric differences over the interval $[-a_{1/(2t)}, a_{1/(2t)}]$, and the tail involves an error of weighted polynomial approximation over the remainder of $(-1, 1)$. For $h > 0$, an interval J, and $r \geqslant 1$, we define the rth symmetric difference as

$$\Delta_h^r(f, x, J) := \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(x + \frac{rh}{2} - ih\right), \tag{1.8}$$

provided all arguments of f lie in J, and 0 otherwise. Sometimes the increment h will depend on x and the function

$$\Phi_t(x) := \sqrt{\left|1 - \frac{|x|}{a_{1/t}}\right|} + T(a_{1/t})^{-1/2}, \qquad x \in (-1, 1). \tag{1.9}$$

This is the case in our modulus of continuity

$$\omega_{r,p}(f, w, t) := \sup_{0<h\leqslant t} \|w\Delta^r_{h\Phi_t(x)}(f, x, (-1,1))\|_{L_p(|x|\leqslant a_{1/(2t)})} + \inf_{P\in\mathscr{P}_{r-1}} \|(f-P)w\|_{L_p(a_{1/(4t)}\leqslant |x|\leqslant 1)} \tag{1.10}$$

and its averaged cousin

$$\bar{\omega}_{r,p}(f, w, t) := \left(\frac{1}{t}\int_0^t \|w\Delta^r_{h\Phi_t(x)}(f, x, (-1,1))\|^p_{L_p(|x|\leqslant a_{1/(2t)})}\, dh\right)^{1/p} + \inf_{P\in\mathscr{P}_{r-1}} \|(f-P)w\|_{L_p(a_{1/(4t)}\leqslant |x|\leqslant 1)}. \tag{1.11}$$

(If $p=\infty$, $\bar{\omega}_{r,p} := \omega_{r,p}$). One can think of $h\Phi_t(x)$ as a suitable replacement for the factor $h\sqrt{1-x^2}$ that appears in the Ditzian–Totik modulus of continuity.

The inf in the tail is at first disconcerting, but note that it is over polynomials of degree at most $r-1$, not n. Its presence ensures that for $f\in\mathscr{P}_{r-1}$, $\omega_{r,p}(f, w, t)\equiv 0$. It also reflects the inability of weighted polynomials $P_n w$ to approximate well beyond the interval $[-a_n, a_n]$. For classical Jacobi weights, the interval $[-a_n, a_n]$ is essentially $[-(1-n^{-2}), 1-n^{-2}]$ and the length of the remaining subintervals of $[-1, 1]$, namely n^{-2}, is negligible. However, for our weights, a_n may be significantly smaller, and the "tail" interval cannot be ignored. For example, for $w_{k,\alpha}$ of (1.5) with $k\geqslant 1$,

$$1 - a_n \sim (\log_k n)^{-1/\alpha},$$

where $\log_k = \log(\log(\cdots(\log(\cdot\cdot)))$ denotes the kth iterated logarithm. Here and in the remainder of the article

$$c_n \sim d_n$$

means that there exist $C_1, C_2 > 0$ such that

$$C_1 \leqslant c_n/d_n \leqslant C_2$$

for the relevant range of n. Similar notation is used for functions and sequence of functions.

We remark that we could probably replace $a_{1/(2t)}$ in the above definition of our moduli of continuity with $a_{1/t} - C_1 t/\sqrt{T(a_{1/t})}$, which is somewhat larger, since, as we shall see in Section 2,

$$a_{1/t} - a_{1/(2t)} \geqslant C_1/T(a_{1/t}) \gg t/\sqrt{T(a_{1/t})}.$$

Likewise, we could probably replace $a_{1/(4t)}$ in the moduli with the somewhat larger $a_{1/t} - C_2 t/\sqrt{T(a_{1/t})}$, with suitably chosen $C_j, j = 1, 2$. However, the resulting moduli are probably equivalent to those above, and the extra complications and hypotheses on the weight are not worth the effort.

The moduli of continuity are rather difficult to assimilate (as is the case with all their cousins [6] for weighted approximation on $\mathbb{R}$). A good way to view the modulus is that for purposes of approximation by polynomials of degree at most n, essentially $t = 1/n$, the main part is taken over the range $[-a_{n/2}, a_{n/2}]$, and the tail is taken over $[-1, 1] \backslash [-a_{n/4}, a_{n/4}]$. Moreover, the function $\Phi_t(x)$ describes the improvement in the degree of approximation in the range $\{x: a_{\alpha n} \leqslant |x| \leqslant a_{n/2}\}$, any fixed $\alpha \in (0, \frac{1}{2})$, in much the same way that $\sqrt{1-x^2}$ does for Jacobi weights on $[-1, 1]$. In particular for x over this range, $\Phi_t(x) \sim T(a_n)^{-1/2} \to 0, n \to \infty$.

Our main result is:

THEOREM 1.2. *Let* $w := e^{-Q} \in \mathscr{E}$. *Let* $r \geqslant 1$ *and* $0 < p \leqslant \infty$. *Then for* $f: (-1, 1) \to \mathbb{R}$ *for which* $fw \in L_p(-1, 1)$ (*and for* $p = \infty$, *we require* f *to be continuous and* fw *to vanish at* ± 1), *we have for* $n \geqslant C_3$

$$E_n[f]_{w,p} \leqslant C_1 \bar{\omega}_{r,p}\left(f, w, \frac{C_2}{n}\right) \leqslant C_1 \omega_{r,p}\left(f, w, \frac{C_2}{n}\right), \tag{1.12}$$

where C_j, $j = 1, 2, 3$, *do not depend on* f *or* n.

We note that the result may be easily extended to hold for $n \geqslant r-1$. For a proof of this for the range $C_3 \geqslant n \geqslant r-1$ in the related case of Freud weights, see [5]. The proof is exactly the same here.

Unfortunately, the moduli above are not obviously monotone increasing in t, so we also present a result involving the increasing modulus

$$\omega^*_{r,p}(f, w, t) := \sup_{\substack{0 < h \leqslant t \\ 0 < \tau \leqslant L}} \|w \Delta^r_{\tau h \Phi_h(x)}(f, x, (-1, 1))\|_{L_p(|x| \leqslant a_{1/(2h)})}$$

$$+ \inf_{P \in \mathscr{P}_{r-1}} \|(f-P)w\|_{L_p(a_{1/(4t)} \leqslant |x| \leqslant 1)}. \tag{1.13}$$

Here L is a (large enough) number independent of f, t.

THEOREM 1.3. *Under the hypotheses of Theorem* 1.2,

$$E_n[f]_{w,p} \leqslant C_1 \omega^*_{r,p}\left(f, w, \frac{C_2}{n}\right), \tag{1.14}$$

$n \geqslant C_3$, *where* $C_j, j=1, 2, 3$, *do not depend on* f *or* n.

The moduli of continuity will be analyzed in part II of this paper, and in particular the relationship to K-functionals/ realization functionals will be discussed. These have the consequence that we can dispense with the constant C_2 inside the moduli in (1.12) but this requires extra hypotheses on w, namely, a Markov–Bernstein inequality.

The paper is organised as follows: In Section 2, we present some technical details related to Q, a_u, and so on. In Section 3, we present estimates involving differences. In Section 4, we obtain polynomial approximations to w^{-1} over suitable intervals, and then in Section 5, we present our crucial approximations to characteristic functions. We prove Theorem 1.2 in Section 6 and Theorem 1.3 in Section 7. Moreover, we discuss some further simplification of the modulus $\omega^*_{r,p}$ in Section 7.

At a first reading, the reader should first read Section 6 and then Sections 4 and 5. The technical Sections 2 and 3 can be read last.

We close this section with a little more notation. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x and $P \in \mathscr{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C \neq C(L)$ to indicate that C is independent of L. Moreover, when dealing with, for example, $x, y \in (C, 1)$, it is taken as understood that $C < 1$. In the sequel, we assume that $w = e^{-Q} \in \mathscr{E}$, except that we shall not use condition (e) of Definition 1.1 unless specified.

2. TECHNICAL LEMMAS

In this section, we shall assume $w \in \mathscr{E}$, except that we shall not use condition (e) of Definition 1.1.

LEMMA 2.1. (a) *For some* $C_j, j=1, 2, 3$, *and* $s \geqslant r \geqslant C_3$,

$$\left(\frac{s}{r}\right)^{C_2 T(r)} \leqslant \frac{Q(s)}{Q(r)} \leqslant \left(\frac{s}{r}\right)^{C_1 T(s)}. \tag{2.1}$$

Moreover,

$$\left(\frac{s}{r}\right)^{C_2 T(r)} \frac{T(s)}{T(r)} \leqslant \frac{Q'(s)}{Q'(r)} \leqslant \frac{T(s)}{T(r)} \left(\frac{s}{r}\right)^{C_1 T(s)}. \tag{2.2}$$

(b) *For some* C_j, $j=1, 2, 3$ *and* $x \in (C_1, 1)$,

$$T(x) \geqslant \frac{C_2}{1-x}; \tag{2.3}$$

$$Q^{(j)}(x) \geqslant \frac{C_2}{(1-x)^{C_3+j}}, \qquad j=0, 1. \tag{2.4}$$

(c) *Given* $\delta > 0$, *there exists* C *such that*

$$T(y) \sim T\left(y\left(1-\frac{\delta}{T(y)}\right)\right), \qquad y \geqslant C. \tag{2.5}$$

Proof. (a) First, (2.1) follows from the fact that for $s \geqslant r \geqslant C_3$,

$$\log \frac{Q(s)}{Q(r)} = \int_r^s T(t)\, dt \sim \int_r^s \frac{T(t)}{t}\, dt$$

and the fact that T is quasi-increasing. Then the identity $Q'(u) = T(u)\, Q(u)$ gives (2.2).

(b) Since Q is increasing, we can assume that $C_2 > 1$ in (1.4). Then writing $C_2 = 1+\delta$, $\delta > 0$, we have

$$\frac{Q'(y)}{Q(y)^{1+\delta}} \leqslant C_1 \frac{Q'(x)}{Q(x)^{1+\delta}}, \qquad y \geqslant x \geqslant C_3.$$

Then as $Q(1) = \infty$, we obtain

$$C_1 \frac{Q'(x)}{Q(x)^{1+\delta}} (1-x) \geqslant \int_x^1 \frac{Q'(y)}{Q(y)^{1+\delta}}\, dy = \frac{1}{\delta Q(x)^{\delta}}$$

so

$$T(x) = \frac{Q'(x)}{Q(x)} \geqslant \frac{C_2}{1-x}.$$

Integrating yields

$$Q(x) \geqslant C_3(1-x)^{-C_2}$$

and so

$$Q'(x) = Q(x)\, T(x) \geqslant C_3(1-x)^{-1-C_2}.$$

(c) We can reformulate (1.4) as

$$\frac{T(y)}{T(x)} \leqslant C_1 \left(\frac{Q(y)}{Q(x)}\right)^{C_2-1}.$$

Hence for $x = y(1-\delta/T(y))$, the quasi-increasing nature of T gives

$$C_4 \leqslant \frac{T(y)}{T(x)} \leqslant C_1 \quad \exp \quad \left((C_2-1)\int_x^y \frac{Q'(t)}{Q(t)}\,dt\right)$$

$$= C_1 \exp\left((C_2-1)\int_x^y t\,\frac{T(t)}{t}\,dt\right) \leqslant C_1 \exp\left(C_5\, T(y) \log \frac{y}{x}\right) \leqslant C_6.$$

Recall here that $T(y)$ is large for y close to 1. ∎

Next, some properties of a_u:

Lemma 2.2. (a) *a_u is uniquely defined and continuous for $u \in (0, \infty)$ and is a strictly increasing function of u.*

(b) *For $u \geqslant C$,*

$$Q'(a_u) \sim u T(a_u)^{1/2}; \tag{2.6}$$

$$Q(a_u) \sim u T(a_u)^{-1/2}. \tag{2.7}$$

(c) *Given fixed $\beta > 0$, we have for large u,*

$$T(a_{\beta u}) \sim T(a_u). \tag{2.8}$$

(d) *Given fixed $\alpha > 1$,*

$$\frac{a_{\alpha u}}{a_u} - 1 \sim \frac{1}{T(a_u)}. \tag{2.9}$$

(e) *If $\alpha > 1$, then for large enough u,*

$$\frac{Q(a_{\alpha u})}{Q(a_u)} \geqslant C_7 > 1. \tag{2.10}$$

(f) *For some* C_j, $j=8, 9, \ldots 12$, $u \geqslant C_8$, *and* $L \geqslant 1$,

$$\exp\left(C_{12}\frac{\log(C_{11}L)}{T(a_u)}\right) \geqslant \frac{a_{Lu}}{a_u} \geqslant 1 + C_{10}\frac{\log(C_9 L)}{T(a_{Lu})}. \tag{2.11}$$

(g) *If* C_2 *is as in* (1.4),

$$T(a_u) \leqslant C_6 u^{2([C_2-1]/[C_2+1])} = C_6 u^{2(1-\delta)} \tag{2.12}$$

some $\delta > 0$.

Proof. (a) The function $u \to a_u$ is the inverse of the strictly increasing continuous function

$$a \to \frac{2}{\pi}\int_0^1 atQ'(at)\frac{dt}{\sqrt{1-t_2}}\,dt, \qquad a \in (0, 1),$$

which has right limit 0 at 0. (Note that this function is continuous even if Q' is not.) We claim also that the function has limit ∞ as $a \to 1-$. For, if $a, t \geqslant C$, (2.4) gives

$$Q'(at) \geqslant C_1(1-at)^{-C_3-1}.$$

So the assertion follows and hence a_u is defined for all $u \in (0, \infty)$.

(b) For u so large that $T(a_u) > 2$, we have

$$\begin{aligned}
\frac{u}{a_u Q'(a_u)} &= \frac{2}{\pi}\left[\int_0^{1-1/T(a_u)} + \int_{1-1/T(a_u)}^1\right]\frac{a_u t Q'(a_u t)}{a_u Q'(a_u)}\frac{dt}{\sqrt{1-t^2}} \\
&\leqslant \frac{2}{\pi}T(a_u)^{1/2}\int_0^{1-1/T(a_u)}\frac{a_u Q'(a_u t)}{a_u Q'(a_u)}\,dt + \frac{2}{\pi}\int_{1-1/T(a_u)}^1\frac{dt}{\sqrt{1-t^2}} \\
&\leqslant \frac{2}{\pi}T(a_u)^{1/2}\frac{Q(a_u)-Q(0)}{a_u Q'(a_u)} + \frac{4}{\pi}T(a_u)^{-1/2} \\
&\leqslant \frac{4}{\pi}T(a_u)^{1/2}\frac{Q(a_u)}{a_u Q'(a_u)} + \frac{4}{\pi}T(a_u)^{-1/2} \leqslant \frac{12}{\pi}T(a_u)^{-1/2}.
\end{aligned}$$

Here we also need u so large that $Q(a_u) \geqslant |Q(0)|$ and $a_u \geqslant \frac{1}{2}$. So we have

$$a_u Q'(a_u) \geqslant \frac{\pi}{12}uT(a_u)^{1/2}.$$

In the other direction, (2.2) gives, for large u,

$$\frac{u}{a_u Q'(a_u)} = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)}{a_u Q'(a_u)} \frac{dt}{\sqrt{1-t^2}}$$

$$\geqslant C_1 \int_{1/2}^1 \frac{T(a_u t)}{T(a_u)} t^{C_1 T(a_u)} \frac{dt}{\sqrt{1-t^2}}$$

$$\geqslant C_2 \frac{T\left(a_u\left(1-\frac{1}{T(a_u)}\right)\right)}{T(a_u)} \left(1-\frac{1}{T(a_u)}\right)^{C_1 T(a_u)} \int_{1-1/T(a_u)}^1 \frac{dt}{\sqrt{1-t^2}}$$

$$\geqslant C_3 T(a_u)^{-1/2}.$$

Here we have used (2.5) and the quasi-monotonicity of T. So we have (2.6). Then (2.7) follows from the definition of T.

(c) We can assume $\beta > 1$. Then by (2.7), and quasi-monotonicity of T,

$$C_1 \leqslant \frac{T(a_{\beta u})}{T(a_u)} \sim \left[\frac{\beta u}{Q(a_{\beta u})}\right]^2 \Big/ \left[\frac{u}{Q(a_u)}\right]^2 \leqslant \beta^2.$$

(d) Now for fixed $\alpha > 1$,

$$\alpha u = \frac{2}{\pi} \int_0^1 a_{\alpha u} t Q'(a_{\alpha u} t) \frac{dt}{\sqrt{1-t^2}}$$

$$\geqslant \frac{2}{\pi} \int_{a_u/a_{\alpha u}}^1 a_u Q'(a_u) \frac{dt}{\sqrt{1-t^2}}$$

$$\geqslant C_2 u T(a_u)^{1/2} \left(1 - \frac{a_u}{a_{\alpha u}}\right)^{1/2}$$

by (2.6). Hence

$$1 - \frac{a_u}{a_{\alpha u}} \leqslant C_3 / T(a_u).$$

In the other direction,

$$\alpha u = \frac{2}{\pi}\left[\int_0^{a_u/a_{\alpha u}} + \int_{a_u/a_{\alpha u}}^1\right] a_{\alpha u} t Q'(a_{\alpha u} t) \frac{dt}{\sqrt{1-t^2}}$$
$$\leqslant \frac{2}{\pi}\int_0^{a_u/a_{\alpha u}} a_{\alpha u} t Q'(a_{\alpha u} t) \frac{dt}{\sqrt{1-\left(\frac{a_{\alpha u} t}{a_u}\right)^2}} + \frac{2}{\pi} a_{\alpha u} Q'(a_{\alpha u}) \int_{a_u/a_{\alpha u}}^1 \frac{dt}{\sqrt{1-t}}$$
$$\leqslant \frac{a_u}{a_{\alpha u}}\left[\frac{2}{\pi}\int_0^1 a_u s Q'(a_u s) \frac{ds}{\sqrt{1-s^2}}\right] + \frac{4}{\pi} a_{\alpha u} Q'(a_{\alpha u}) \left(1-\frac{a_u}{a_{\alpha u}}\right)^{1/2}$$
$$\leqslant u + CuT(a_u)^{1/2}\left(1-\frac{a_u}{a_{\alpha u}}\right)^{1/2}$$

by (2.6) and (2.8). Then

$$1-\frac{a_u}{a_{\alpha u}} \geqslant \left(\frac{\alpha-1}{C}\right)^2 \frac{1}{T(a_u)}.$$

(e) For large enough u,

$$\frac{Q(a_{\alpha u})}{Q(a_u)} = \exp\left(\int_{a_u}^{a_{\alpha u}} t\,\frac{T(t)}{t}\,dt\right)$$
$$\geqslant \exp\left(C_6 T(a_u) \log\left(\frac{a_{\alpha u}}{a_u}\right)\right) \geqslant \exp(C_7) > 1,$$

by (d) of this lemma.

(f) From (1.4) with $y = a_{Lu}$ and $x = a_u$,

$$\frac{T(a_{Lu})}{T(a_u)} \leqslant C\left(\frac{Q(a_{Lu})}{Q(a_u)}\right)^{C_2-1}. \tag{2.13}$$

This forces $C_2 > 1$, as the left-hand side $\to\infty$ as $L\to\infty$. Then with the constants in $\sim$ independent of L, (2.7) gives

$$\frac{Q(a_{Lu})}{Q(a_u)} \sim \frac{LuT(a_{Lu})^{-1/2}}{uT(a_u)^{-1/2}}$$
$$\geqslant CL\left(\frac{Q(a_{Lu})}{Q(a_u)}\right)^{-(C_2-1)/2} \qquad \text{(by (2.13))}$$
$$\Rightarrow \frac{Q(a_{Lu})}{Q(a_u)} \geqslant CL^{2/(1+C_2)}.$$

Then using (2.1),

$$\left(\frac{a_{Lu}}{a_u}\right)^{C_1 T(a_{Lu})} \geqslant CL^{2/(1+C_2)}$$

and the right inequality in (2.11) follows if we use $u-1 \geqslant \log u$, $u \geqslant 1$. In the other direction, (2.1) and then (2.7) give

$$\begin{aligned}\frac{a_{Lu}}{a_u} &\leqslant \left(\frac{Q(a_{Lu})}{Q(a_u)}\right)^{1/(C_2 T(a_u))} \\ &\leqslant \left(C_1 \frac{LuT(a_{Lu})^{-1/2}}{uT(a_u)^{-1/2}}\right)^{1/(C_2 T(a_u))} \leqslant (C_3 L)^{1/(C_2 T(a_u))}.\end{aligned}$$

Here the constants are independent of L and u. Then the left inequality in (2.11) follows.

(g) We apply (1.4) with $y = a_u$ and $x = C_3$, so that

$$\begin{aligned}&Q'(a_u) \leqslant C_4 Q(a_u)^{C_2} \\ &\Rightarrow uT(a_u)^{1/2} \leqslant C_5 (uT(a_u)^{-1/2})^{C_2}.\end{aligned}$$

Rearranging this gives (2.12). ∎

We finish this section with an infinite finite-range inequality: We provide a proof, as those in the literature [8, 10, 12–14, 18] do not quite match our needs/hypotheses:

Lemma 2.3. *Let $0 < p \leqslant \infty$, $s > 1$. Then for some C_1, $C_2 > 0$, $n \geqslant 1$, and $P \in \mathcal{P}_n$,*

$$\|Pw\|_{L_p(-1,1)} \leqslant C_1 \|Pw\|_{L_p(-a_{sn}, a_{sn})}. \tag{2.14}$$

Moreover,

$$\|Pw\|_{L_p(|x| \geqslant a_{sn})} \leqslant C_1 e^{-C_2 nT(a_n)^{-1/2}} \|Pw\|_{L_p(-a_{sn}, a_{sn})}. \tag{2.15}$$

Remark. Note that (2.12) shows that for some $C_3 > 0$, and large enough n,

$$nT(a_n)^{-1/2} \geqslant n^{C_3}.$$

Proof. We may change Q in a closed subinterval of $(-1, 1)$ without affecting (2.14), (2.15) apart from increasing the constants. Note too that the affect on a_u is marginal and is absorbed into the fact that $s > 1$. Thus we may assume that Q' is continuous in $(-1, 1)$. This and the strict monotonicity of $tQ'(t)$ in $(0, 1)$ allow us to apply existing sup-norm inequalities to deduce that for $P \in \mathcal{P}_n$,

$$\|Pw\|_{L_\infty(-1,1)} \leqslant C \|Pw\|_{L_\infty[-a_n, a_n]}.$$

For a precise reference, see [8, Theorem 4.5], for example. Moreover, the proof of Lemma 6.1 in [10, pp. 57–58] gives without change

$$|Pw|^p (a_n x) \leqslant \frac{1}{\pi} \frac{2x}{x-1} \int_{-1}^{1} |Pw|^p (a_n t)\, dt, \qquad x \in (1, 1/a_n). \qquad (2.16)$$

Let $\langle x \rangle$ denote the greatest integer $\leqslant x$. Let δ be small and positive, let $l := \langle \delta n \rangle$ and let $T_l(x)$ denote the Chebyshev polynomial of degree l. Using the identity

$$T_l(x) = \tfrac{1}{2}[(x + \sqrt{x^2-1})^l + (x - \sqrt{x^2-1})^l], \qquad x > 1, \qquad (2.17)$$

it is not difficult to see that

$$T_l(x) \geqslant \frac{1}{2} \exp\left(\frac{l}{\sqrt{2}} \sqrt{x-1}\right), \qquad x \in \left(1, \frac{9}{8}\right). \qquad (2.18)$$

We now let $m := n + l = n + \langle \delta n \rangle$, $m' := n + 2l = n + 2\langle \delta n \rangle$ and apply (2.16) to $P(x)\, T_l(x/a_m) \in \mathcal{P}_m$. We obtain for $x > 1$,

$$|Pw|^p (a_m x) \leqslant T_l(x)^{-p} \frac{1}{\pi} \frac{2x}{x-1} \int_{-1}^{1} |Pw|^p (a_m t)\, dt.$$

Replacing $a_m x$ by y and integrating from $a_{m'}$ to 1 gives

$$\int_{a_{m'}}^{1} |Pw|^p (y)\, dy \leqslant \left(\int_{-a_m}^{a_m} |Pw|^p (s)\, ds\right)\left(\frac{2}{\pi} \int_{a_{m'}}^{1} \frac{y}{y - a_m} T_l\left(\frac{y}{a_m}\right)^{-p} \frac{dy}{a_m}\right).$$

Here using (2.18),

$$\begin{aligned}
\int_{a_{m'}}^{1} \frac{y}{y-a_m} T_l\left(\frac{y}{a_m}\right)^{-p} \frac{dy}{a_m} &= \int_{a_{m'}/a_m}^{1/a_m} \frac{x}{x-1} T_l(x)^{-p}\, dx \\
&\leqslant C\left(\int_{a_{m'}/a_m}^{9/8} \frac{1}{x-1} \exp\left(-\frac{lp}{\sqrt{2}}\sqrt{x-1}\right) dx\right) \\
&\leqslant C_1 \log\left(\frac{9/8}{a_{m'}/a_m - 1}\right) \exp\left(-C_2 l \left(\frac{a_{m'}}{a_m} - 1\right)^{1/2}\right) \\
&\leqslant C_3 \exp(-C_4 n T(a_n)^{-1/2}).
\end{aligned}$$

Here we have used (2.9) and our choice of l. Now if δ is small enough, $m' \leqslant sn$. Then (2.15) follows easily and in turn yields (2.14). ∎

3. TECHNICAL LEMMAS ON Φ_t

In this section, we present various estimates involving the function $\Phi_t(x)$. Throughout, we assume that $w = e^{-Q} \in \mathscr{E}$, except that we do not assume (e) of Definition 1.1. Our first lemma concerns the function

$$\Phi_t(x) = \sqrt{\left|1 - \frac{|x|}{a_{1/t}}\right|} + T(a_{1/t})^{-1/2}, \qquad x > 0.$$

LEMMA 3.1. (a) *There exist* C_1, C_2 *independent of* s, t, x, *such that for* $0 < t < s \leqslant C_1$,

$$\Phi_s(x) \leqslant C_2 \Phi_t(x), \quad |x| \leqslant a_{1/s}. \tag{3.1}$$

(b) *There exists* C_1 *such that for* $0 < s \leqslant C_1$ *and* $s/2 \leqslant t \leqslant s$,

$$\Phi_s(x) \sim \Phi_t(x), \quad |x| < 1. \tag{3.2}$$

Proof. (a) Let $\delta > 0$ be fixed. First for

$$|x| \leqslant a_{1/s}(1 - \delta/T(a_{1/s})) \Leftrightarrow 1 - |x|/a_{1/s} \geqslant \delta/T(a_{1/s})$$

we have

$$\Phi_s(x) \sim \sqrt{1 - \frac{|x|}{a_{1/s}}} \leqslant \sqrt{1 - \frac{|x|}{a_{1/t}}} \leqslant \Phi_t(x).$$

Next, for

$$a_{1/s}(1 - \delta/T(a_{1/s})) \leqslant |x| \leqslant a_{1/s} \Rightarrow 1 - |x|/a_{1/s} \leqslant \delta/T(a_{1/s})$$

we have

$$\Phi_s(x) \sim T(a_{1/s})^{-1/2}.$$

This is bounded by $C\Phi_t(x)$ if $|1 - |x|/a_{1/t}| \geqslant \delta/T(a_{1/s})$, for a fixed $\delta > 0$. Otherwise, we have $|1 - |x|/a_{1/s}| \leqslant \delta/T(a_{1/s})$ and $|1 - |x|/a_{1/t}| \leqslant \delta/T(a_{1/s})$, so

$$\left|1 - \frac{a_{1/t}}{a_{1/s}}\right| = \left|\left(1 - \frac{|x|}{a_{1/s}}\right) - \frac{|x|}{a_{1/s}}\left(\frac{a_{1/t}}{|x|} - 1\right)\right|$$

$$\leqslant C_1 \delta / T(a_{1/s}).$$

If δ is small enough, we deduce from (2.9) and (2.8) that

$$T(a_{1/t}) \sim T(a_{1/s}),$$

so

$$\Phi_s(x) \sim T(a_{1/s})^{-1/2} \sim T(a_{1/t})^{-1/2} \sim \Phi_t(x)$$

and again (3.1) follows.

(b) Now

$$\left|1-\frac{|x|}{a_{1/t}}\right| = \left|1-\frac{|x|}{a_{1/s}}+\frac{|x|}{a_{1/s}}\left(1-\frac{a_{1/s}}{a_{1/t}}\right)\right|$$
$$\leqslant \left|1-\frac{|x|}{a_{1/s}}\right| + O\left(\frac{1}{T(a_{1/s})}\right).$$

Then we obtain for $|x|<1$

$$\Phi_t(x) \leqslant C\Phi_s(x).$$

The converse direction is similar. ∎

LEMMA 3.2. (a) *Let $L>0$. Uniformly for $u \geqslant 1$, and $|x|$, $|y| \leqslant a_u$, such that*

$$|x-y| \leqslant \frac{L}{u}\sqrt{\left|1-\frac{|y|}{a_u}\right|}, \tag{3.3}$$

we have

$$w(x) \sim w(y) \tag{3.4}$$

and

$$1-\frac{|x|}{a_{2u}} \sim 1-\frac{|y|}{a_{2u}}. \tag{3.5}$$

(b) *Let $L>0$. For $t \in (0, t_0)$, $|x|$, $|y| \leqslant a_{1/(Lt)}$ such that*

$$|x-y| \leqslant Lt\Phi_t(x), \tag{3.6}$$

we have (3.4) *and*

$$\Phi_t(x) \sim \Phi_t(y). \tag{3.7}$$

Proof. (a) It suffices to prove (3.4), (3.5) for large u. Moreover, (3.4) and (3.5) are immediate for $|x| \leqslant C < 1$ and large u. Let us suppose that $C \leqslant x \leqslant y \leqslant x + (L/u)\sqrt{|1 - |y|/a_u|}$. Then as $Q'(s)$ is quasi-increasing for s close to 1,

$$0 \leqslant Q(y) - Q(x) \leqslant C_1 Q'(y)(y - x).$$

We have then (3.4) for

$$Q'(y)(y - x) = O(1). \tag{3.8}$$

We shall show that

$$Q'(y)\sqrt{\left|1 - \frac{y}{a_u}\right|} \leqslant C_2 u, \tag{3.9}$$

so that (3.3) implies (3.8) and hence (3.4). If first, $0 < y \leqslant a_u/2$, then

$$Q'(y)\sqrt{\left|1 - \frac{y}{a_u}\right|} \leqslant C_3 Q'(y) \leqslant C_4 a_u Q'(y) \int_{1/2}^{1} \frac{dt}{\sqrt{1 - t^2}}$$
$$\leqslant C_5 \int_{1/2}^{1} a_u t Q'(a_u t) \frac{dt}{\sqrt{1 - t^2}} \leqslant C_6 u.$$

If, on the other hand, $a_u/2 \leqslant y \leqslant a_u$,

$$Q'(y)\sqrt{\left|1 - \frac{y}{a_u}\right|} \leqslant C_7 \int_{y/a_u}^{1} a_u t Q'(a_u t) \frac{dt}{\sqrt{1 - t^2}} \leqslant C_8 u.$$

So we have (3.9) in all cases and hence (3.4). We proceed to prove (3.5). Now from (3.3) and as $y \leqslant a_u$,

$$1 \leqslant \frac{1 - x/a_{2u}}{1 - y/a_{2u}} = 1 + \frac{y - x}{a_{2u}(1 - y/a_{2u})} = 1 + O\left(\frac{1}{u\sqrt{1 - y/a_{2u}}}\right)$$
$$= 1 + O\left(\frac{1}{u\sqrt{1 - a_u/a_{2u}}}\right) = 1 + O\left(\frac{T(a_u)^{1/2}}{u}\right) = 1 + o(1),$$

by (2.9) and (2.12).

(b) Write $Lt = 1/u$, so that $|x|, |y| \leqslant a_{1/(Lt)} = a_u$, and we can recast (3.6) as

$$|x - y| \leqslant C_1 \frac{1}{u}\left[\sqrt{1 - \frac{|x|}{a_u}} + T(a_u)^{-1/2}\right] \leqslant C_2 \frac{1}{2u}\sqrt{1 - \frac{|x|}{a_{2u}}}$$

by (2.8), (2.9), and (3.2). Then (a) gives (3.4), and (3.7) follows easily from (3.5). ∎

4. POLYNOMIAL APPROXIMATION OF w^{-1}

The result of this section is:

THEOREM 4.1. *Assume $w = e^{-Q} \in \mathscr{E}$. For $n \geqslant 1$, there exist polynomials G_n of degree at most C_n, such that*

$$0 \leqslant G_n(x) \leqslant w^{-1}(x), \; x \in (-1, 1) \tag{4.1}$$

and

$$G_n(x) \sim w^{-1}(x), \; |x| \leqslant a_n. \tag{4.2}$$

We remark that this does not follow from existing results in the literature on approximation by weighted polynomials of the form $P_n(x)\, w(a_n x)$ [19] as our weights do not satisfy their hypotheses. The methods of Totik [19] can be applied to give sharper results but we base our proof on a method involving entire functions. It is only in the following result that we need condition (e) of Definition 1.1.

LEMMA 4.2. *There exists an even function*

$$G(z) = \sum_{j=0}^{\infty} g_j z^{2j}, \qquad g_j \geqslant 0 \; \forall j, \tag{4.3}$$

analytic in $\{z: |z| \leqslant 1\}$, such that

$$G(x) \sim w^{-1}(x), \qquad x \in (-1, 1). \tag{4.4}$$

Proof. This is different from that in [10, p. 107ff] because of the different hypotheses on Q, so we include the details. Consider the transformation

$$x := x(r) := \sqrt{\frac{r}{r+1}}, \; r \in (0, \infty).$$

This is equivalent to

$$r = \frac{x^2}{1 - x^2}, \qquad x \in (0, 1).$$

Set

$$Q_1(r) := Q(x(r)) = Q\left(\sqrt{\frac{r}{r+1}}\right), \qquad r \in (0, \infty).$$

We shall apply a theorem of Clunie–Kövari [1] to

$$\phi(r) := e^{Q_1(r)}.$$

Straightforward calculations show that

$$x'(r) = \frac{1}{2x(r)(r+1)^2}$$

and

$$Q_1'(r) = \frac{Q'(x(r))}{2x(r)(r+1)^2}.$$

Next,

$$r+1 = \frac{1}{1-x(r)^2},$$

so if δ is as in Definition 1.1(e),

$$(r+1)^{1-\delta}\, Q_1'(r) = (r+1)^{-1-\delta}\, \frac{Q'(x(r))}{2x(r)} = (1-x(r)^2)^{1+\delta}\, \frac{Q'(x(r))}{2x(r)}$$

is quasi-increasing for large r. Now set

$$\psi(r) := rQ_1'(r).$$

By the quasi-increasing nature of $r^{1-\delta}Q_1'(r)$, we have for large enough λ and some C independent of λ,

$$\begin{aligned}\psi(\lambda r) - \psi(r) &= \{(\lambda r)^{\delta}\,(\lambda r)^{1-\delta}\, Q_1'(\lambda r) - rQ_1'(r)\}\\ &\geqslant r^{1-\delta} Q_1'(r)\{(\lambda r)^{\delta}\, C - r^{\delta}\} \geqslant 1\end{aligned}$$

if λ is large enough, and $r \geqslant r_0$. Moreover,

$$\psi(r) = \frac{rQ'(x(r))}{2x(r)(r+1)^2} = \frac{x(r)\, Q'(x(r))}{2(r+1)} = \frac{x(r)}{2}\, Q'(x(r))(1-x(r)^2)$$

is increasing in r for large r, since $x(r)$ and $Q'(x(r))(1-x(r)^2)$ are. Moreover, $\phi(r) := e^{Q_1(r)}$ admits the representation

$$\phi(r) = \phi(1) \exp\left(\int_1^r \frac{\psi(s)}{s}\, ds\right), \qquad r \geqslant 1.$$

By a theorem of Clunie and Kövari [1, Theorem 4, p. 19], there exists entire

$$H(r) = \sum_{j=0}^{\infty} h_j r^j, \qquad h_j \geqslant 0\ \forall j$$

such that

$$H(r) \sim \phi(r) = \exp\left(Q\left(\sqrt{\frac{r}{r+1}}\right)\right), \qquad r > r_0.$$

Then assuming $h_0 > 0$ as we can, we see that this holds for $r \geqslant 0$. Then

$$G(x) := H\left(\frac{x^2}{1-x^2}\right) = H(r) \sim \exp\left(Q\left(\sqrt{\frac{r}{r+1}}\right)\right) = \exp(Q(x))$$

satisfies (4.4) and as

$$G(x) = H\left(\sum_{j=1}^{\infty} x^{2j}\right)$$

we also obtain (4.3). ∎

Proof of Theorem 4.1. Let J be a positive even integer (to be chosen large enough later) and let $T_n(x)$ denote the classical Chebyshev polynomial on $[-1, 1]$. Let G_n denote the Lagrange interpolant to G at the zeros of $T_n(x/a_n)^J$ so that G_n has degree at most $Jn-1$ and admits the error representation

$$(G - G_n)(x) = \frac{1}{2\pi i} \int_\Gamma \frac{G(t)}{t-x} \left(\frac{T_n(x/a_n)}{T_n(t/a_n)}\right)^J dt$$

for x inside Γ. We shall choose Γ to be the ellipse with foci at $\pm a_n$, intersecting the real and imaginary axes at $(a_n/2)(\rho+\rho^{-1})$ and $(a_n/2)(\rho-\rho^{-1})$, respectively. Here we shall choose for some fixed small $\varepsilon > 0$,

$$\rho := 1 + \left(\frac{\varepsilon}{T(a_n)}\right)^{1/2}.$$

Since G has non-negative Maclaurin series coefficients and satisfies (4.4), we deduce that

$$\delta_n := \|1 - G_n/G\|_{L_\infty[-a_n, a_n]} \leqslant C_1 \frac{w^{-1}((a_n/2)(\rho+\rho^{-1}))}{(\rho-1)^2} \frac{1}{\min_{t \in \Gamma} |T_n(t/a_n)|^J}.$$

Now for $t \in \Gamma$, we can write $t = (a_n/2)(z + z^{-1})$, where $|z| = \rho$, so that

$$\begin{aligned} |T_n(t/a_n)| &= |T_n(\tfrac{1}{2}(z+z^{-1}))| = |\tfrac{1}{2}(z^n + z^{-n})| \\ &\geqslant \tfrac{1}{2}(\rho^n - \rho^{-n}) \geqslant \exp(C_2 n T(a_n)^{-1/2}). \end{aligned}$$

(Recall that $nT(a_n)^{-1/2} \to \infty$ as $n \to \infty$ and in fact grows faster than a power of n). It is important here that C_2 is independent of J. Next

$$\frac{a_n}{2}(\rho + \rho^{-1}) \leqslant a_n \left(1 + C_3 \frac{\varepsilon}{T(a_n)}\right) \leqslant a_{2n}$$

if ε is small enough, and n is large enough, by (2.9). Then

$$w^{-1}\left(\frac{a_n}{2}(\rho+\rho^{-1})\right) \leqslant w^{-1}(a_{2n}) \leqslant \exp(C_4 n T(a_n)^{-1/2}),$$

where again it is important that C_4 is independent of J. Since $(\rho-1)^{-2} \sim T(a_n)$ grows no faster than a power of n, we see that choosing J large enough gives

$$\delta_n \leqslant C T(a_n) \exp(nT(a_n)^{-1/2}(C_4 - C_2 J)) \to 0, \qquad n \to \infty.$$

Then (4.4) gives (4.2).

We now turn to proving (4.1). It suffices to prove

$$0 \leqslant G_n \leqslant C w^{-1}$$

for then (4.1) follows on multiplying G_n by a suitable constant (and (4.2) is still valid)). First, we can assume n is even (for odd n, we can use G_{n+1}) so that $H_n(x) := G_n(\sqrt{x})$ is a polynomial of degree at most $Jn/2 - 1$ (recall that T_n and J are even) that interpolates to the function $H(x) := G(\sqrt{x})$, which is analytic in $(-1, 1)$, at the $Jn/2$ zeros of $T_n(\sqrt{t}/a_n)^J$ that lie in $(0, a_n^2)$. Thus $H_n(x)$ is determined entirely by interpolation conditions. Let γ_n denote the leading coefficient of $T_n(x/\sqrt{a_n})$. Then the usual derivative-error formula for Hermite interpolation gives for $x \in (0, \infty)$ and some $\xi \in (0, 1)$

$$(H - H_n)(x) = \gamma_n^{-J} T_n\left(\frac{\sqrt{x}}{a_n}\right)^J \frac{H^{(Jn/2)}(\xi)}{(Jn/2)!} \geqslant 0.$$

(Recall that H is analytic and has non-negative Maclaurin series coefficients.) So in $(-1, 1)$,

$$G_n \leqslant G \leqslant Cw^{-1}.$$

To show that $G_n \geqslant 0$ in $(-1, 1)$, we note that it is true in $[-a_n, a_n]$ (this follows from (4.2)) and we must establish it elsewhere. We use a zero counting lemma used to prove the Posse–Markov–Stieltjes inequalities [7, p. 30, Lemma 5.3] (there the proof is for $(-\infty, \infty)$, but the proof goes through for $(0, 1)$ with trivial changes). Now H is absolutely monotone in $(0, 1)$ and $H - H_n$ has $Jn/2$ zeros in $(0, a_n^2]$. If m is the number of zeroes of $H_n(x)$ in $[a_n^2, 1)$, Lemma 5.3 in [7, p. 30] gives

$$\frac{Jn}{2} + m \leqslant \deg(H_n) + 1 \leqslant \frac{Jn}{2}.$$

So $m = 0$, that is, H_n has no zeros in $(a_n^2, 1)$. Thus $H_n \geqslant 0$ there, so $G_n \geqslant 0$ in $(-1, 1)$. ∎

5. POLYNOMIALS APPROXIMATING CHARACTERISTIC FUNCTIONS

Our Jackson theorem is based on polynomial approximations to the characteristic function $\chi_{[a, b]}$ of an interval $[a, b]$. We believe the following result is of independent interest:

THEOREM 5.1. *Let l be a positive integer. There exist C_1, J, n_0 such that for $n \geqslant n_0$ and $\tau \in [-a_n, a_n]$, there exist polynomials $R_{n,\tau}$ of degree at most $2lJn$ such that for $x \in (-1, 1)$,*

$$|\chi_{[\tau, a_n]} - R_{n,\tau}|\,(x)\, w(x)/w(\tau) \leqslant C_1 \left(1 + \frac{n\,|x-\tau|}{\sqrt{1 - |\tau|/a_{2n}}}\right)^{-l}. \tag{5.1}$$

We emphasise that the constants are independent of n, τ, x. Our proof will use polynomials from [9] built on the Chebyshev polynomials:

LEMMA 5.2. *There exist C_1, B, n_1, such that for $n \geqslant n_1$ and $|\zeta| \leqslant \cos \pi/2n$, there exists a polynomial $V_{n,\zeta}$ of degree at most $n-1$ with*

$$\|V_{n,\zeta}\|_{L_\infty[-1,1]} = V_{n,\zeta}(\zeta) = 1; \tag{5.2}$$

$$|V_{n,\zeta}(t)| \leqslant \frac{B\sqrt{1-|\zeta|}}{n\,|t-\zeta|}, \qquad t \in (-1, 1)\backslash\{\zeta\}. \tag{5.3}$$

Moreover,

$$V_{n,\zeta}(t) \geq \frac{1}{2}, \; |t-\zeta| \leq C_1 \frac{\sqrt{1-|\zeta|}}{n}. \tag{5.4}$$

The constants are independent of n, ζ, t.

Proof. The assertions (5.2), (5.3) are Proposition 13.1 in [9]. The estimate (5.4) follows from the classical Bernstein inequality. ∎

The polynomials $R_{n,\tau}$ are determined as follows: Let us suppose that, say,

$$a_1 \leq \tau \leq a_n.$$

Later on, we shall suppose that τ exceeds a fixed positive constant. We define

$$\zeta := \frac{\tau}{a_{2lJn}} \tag{5.5}$$

and if G_n are the polynomials of Theorem 4.1,

$$R_{n,\tau}(x) := \frac{\int_0^x G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{\int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}. \tag{5.6}$$

The parameter $\tau^* > \tau$ and J are defined as follows: Let M denote a positive constant such that for, say, $u \geq u_0$,

$$Q'(x) \leq MQ'(a_u), \qquad \tfrac{1}{2} \leq x \leq a_{2u}. \tag{5.7}$$

The existence of such an M follows from (2.6), (2.8). We set

$$H := H(n, \tau, l) := \frac{4ln}{a_n Q'(\tau)\sqrt{1-\zeta}} \tag{5.8}$$

and if $\tau = a_r$,

$$\tau^* := \tau^*(n, \tau) := \min\left\{a_{2r}, a_n, \tau + 2\frac{a_n}{n}\sqrt{1-\zeta}\, H \log H\right\}. \tag{5.9}$$

The reason for this (complicated!) choice will become clearer later. We assume that $J \geq 4$ is so large that G_n has degree at most $Jn-1$, and also

$$J \geq 32M, \tag{5.10}$$

where M is as above. Note that then $R_{n,\tau}$ has degree at most $Jn + lJn \leq 2lJn$. We first record some estimates of the terms in (5.6):

LEMMA 5.3. (a) *For* $n \geqslant n_1$, *and* $C_1 \leqslant \tau \leqslant a_n$, *we have*

$$w(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds \geqslant \frac{C_2}{n}\sqrt{1-\zeta}, \tag{5.11}$$

where $C_2 \neq C_2(n, \tau)$.

(b) *For* $x \in (\tau, a_{2lJn})$,

$$\int_x^{a_{2lJn}} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ/2} ds \leqslant \frac{C_1}{n}\sqrt{1-\zeta}\left(1+\frac{n\,|x-\tau|}{\sqrt{1-\zeta}}\right)^{-l} \tag{5.12}$$

and for $x \in (-a_{2lJn}, \tau)$,

$$\int_{-a_{2lJn}}^{x} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ/2} ds \leqslant \frac{C_1}{n}\sqrt{1-\zeta}\left(1+\frac{n\,|x-\tau|}{\sqrt{1-\zeta}}\right)^{-l}. \tag{5.13}$$

Here $C_1 \neq C_1(n, \tau)$.

Proof. (a) Let us denote the left-hand side of (5.11) by Γ. By (4.2) and (5.4),

$$\Gamma \geqslant C_2 w(\tau) \int_{\tau-(C_3/n)\sqrt{1-\zeta}}^{\tau} w^{-1}(s)\, ds \geqslant \frac{C_4}{n}\sqrt{1-\zeta},$$

where we have used (3.4) of Lemma 3.2(a).

(b) These follow in a straightforward fashion from the estimates (5.2), (5.3) and the fact that $J \geqslant 4$. ∎

Now we begin the proof of Theorem 5.1. We first show that it suffices to consider τ in the range $[S, a_n]$ for some fixed $S<1$.

Proof of Theorem 5.1 *for* $|\tau| \leqslant S$, *where* $S<1$ *is fixed.* Note first that since for such τ,

$$w(x)/w(\tau) \leqslant w(0)/w(S), \qquad x \in (-1, 1),$$

we must only prove there exists $R_{n,\tau}$ of degree at most n such that

$$|\chi_{[\tau, a_n]} - R_{n,\tau}|\,(x) \leqslant C_1 \left(1+\frac{n\,|x-\tau|}{\sqrt{1-\frac{|\tau|}{a_{2n}}}}\right)^{-l},$$

for $|x| \leqslant 1$. Setting here $\xi := \tau/a_n$, and $s := x/a_n$, and $U_{n,\xi}(s) := R_{n,\tau}(x) = R_{n,\tau}(a_n s)$, we see that it suffices to show that

$$|\chi_{[\xi,1]}(s) - U_{n,\xi}(s)| \leqslant C_2(1 + n|s - \xi|)^{-l}, \qquad s \in [-2, 2].$$

We have used here that $|\xi| \leqslant C < 1$, for large n. The existence of such polynomials is classical. See for example [4]. One could also base them on the $V_{n,\zeta}$ above. ∎

It suffices to consider $\tau \in [S, a_n]$, *where* S *is fixed.* For, once this is done, we have the result for all $\tau \in [0, a_n]$. With the result for $\tau \geqslant 0$, we set

$$R_{n,-\tau}(x) := 1 - R_{n,\tau}(-x), \qquad x \in (-1, 1).$$

It is not difficult to check the result for $-\tau$ from the corresponding result for τ, using the identity

$$\chi_{[-\tau, a_n]}(x) = 1 - \chi_{(\tau, a_n]}(-x). \quad ∎$$

In the sequel, we define $R_{n,\tau}$ by (5.6)–(5.10).

It suffices to prove (5.1) *for* $\tau \in [S, a_n]$ *and* $|x| \leqslant a_{2lJn}$. For then (5.1) for this restricted range implies

$$\left\| \left(1 + \left[\frac{n(x-\tau)}{\sqrt{1-\tau/a_{2n}}}\right]^2\right)^l R_{n,\tau}(x)\, \frac{w(x)}{w(\tau)} \right\|_{L_\infty[-a_{2lJn},\, a_{2lJn}]} \leqslant C_3 n^{C_4},$$

where $C_4 \neq C_4(n, \tau)$. Since the polynomial on the left-hand side has degree at most $2l + Jn + lJn \leqslant \eta 2lJn$, some fixed $\eta < 1$, if $l \geqslant 2$ and n is large enough (as we can assume), then the infinite-finite range inequality Lemma 2.3 gives

$$\left\| \left(1 + \left[\frac{n(x-\tau)}{\sqrt{1-\tau/a_{2n}}}\right]^2\right)^l R_{n,\tau}(x)\, \frac{w(x)}{w(\tau)} \right\|_{L_\infty(a_{2lJn} \leqslant |x| \leqslant 1} \leqslant C_5 \exp(-n^{C_6}).$$

Then (5.1) follows for $|x| \geqslant a_{2lJn}$. ∎

We can now begin the proof of (5.1) proper. We consider five different ranges of x: $[0, \tau)$, $[\tau, \tau^*]$, $(\tau^*, a_n]$, $(a_n, a_{2lJn}]$, $[-a_{2lJn}, 0)$. Moreover, we set

$$\Delta(x) := |\chi_{[\tau, a_n]} - R_{n,\tau}|\,(x)\, w(x)/w(\tau).$$

Proof of (5.1) *for* $x \in [0, \tau)$. Here using (4.1), and then (5.11),

$$\Delta(x) = \frac{w(x) \int_0^x G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{w(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}$$

$$\leqslant C \frac{w(x) \int_0^x w^{-1}(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{(1/n) \sqrt{1-\zeta}}$$

$$\leqslant C \frac{\int_0^x V_{n,\zeta}\,(s/a_{2lJn})^{lJ}\, ds}{(1/n) \sqrt{1-\zeta}}$$

by the monotonicity of w. Then (5.13) gives the result. Note that uniformly in τ and n,

$$1-\zeta = 1 - \frac{\tau}{a_{2lJn}} \sim 1 - \frac{\tau}{a_{2n}}. \quad \blacksquare$$

Proof of (5.1) *for* $x \in [\tau, \tau^*)$. Here

$$\Delta(x) = \frac{w(x) \int_x^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{w(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}$$

$$\leqslant C \frac{\int_x^{\tau^*} \exp(Q(s) - Q(x))\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{(a_n/n) \sqrt{1-\zeta}}$$

by (4.1) and (5.11). Now for $s \in (x, \tau^*)$, the property (5.7) of Q' gives (recall $\tau = a_r$ and $\tau^* \leqslant a_{2r}$)

$$Q(s) - Q(x) \leqslant MQ'(a_r)(s-x) \leqslant MQ'(\tau)(s-\tau).$$

Then using our bounds on $V_{n,\zeta}$ in (5.2), (5.3), we have

$$\Delta(x) \leqslant C_1 \frac{\int_x^{\tau^*} \exp(MQ'(\tau)(s-\tau)) \min\{1, Ba_{2lJn} \sqrt{1-\zeta}/(n(s-\tau))\}^{lJ}\, ds}{(a_{2lJn}/n) \sqrt{1-\zeta}}$$

$$= C_1 B \int_{n(x-\tau)/Ba_{2lJn}\sqrt{1-\zeta}}^{n(\tau^*-\tau)/Ba_{2lJn}\sqrt{1-\zeta}} \exp\left(\frac{a_{2lJn}}{a_n} \frac{4lMBu}{H}\right) \min\left\{1, \frac{1}{u}\right\}^{lJ} du$$

$$\leqslant C_2 \int_{n(x-\tau)/Ba_{2lJn}\sqrt{1-\zeta}}^{(2/B)\, H \log H} g(u) \min\left\{1, \frac{1}{u}\right\}^{lJ/2} du$$

for say $n \geqslant n_1 = n_1(J, L)$ by (5.9) and where

$$g(u) := \exp\left(\frac{8lMBu}{H}\right) \min\left\{1, \frac{1}{u}\right\}^{lJ/2}.$$

We claim that if J is large enough,

$$g(u) \leqslant C_3, \; u \in \left[0, \frac{2}{B} H \log H\right],$$

with C_3 independent of τ, n. First we show that

$$H \geqslant e; \; H \geqslant e^{B/2} \tag{5.14}$$

uniformly for $\tau \in [S, a_n]$ and $n \geqslant n_0(J, l)$. Recall that B, J, M are independent of l (see (5.3), (5.7), (5.10)). Next, from (3.9), for $\tau \in [S, a_n]$,

$$Q'(\tau) \sqrt{1 - \frac{\tau}{a_{2n}}} \leqslant C_4 n,$$

with $C_4 \neq C_4(n, \tau, l)$. Then from (5.8),

$$H \geqslant \frac{4l}{C_4} \left(\frac{1 - \tau/a_{2n}}{1 - \tau/a_{2lJn}} \right)^{1/2}.$$

Here for $n \geqslant n_0(J, l)$, we see using the inequality $1 - u \leqslant \log(1/u)$, $u \in (0, 1]$, we obtain

$$\begin{aligned} \frac{1 - \tau/a_{2lJn}}{1 - \tau/a_{2n}} &= 1 + \frac{\tau}{a_{2n}} \frac{1 - a_{2n}/a_{2lJn}}{1 - \tau/a_{2n}} \\ &\leqslant 1 + \frac{\log (a_{2lJn}/a_{2n})}{1 - a_n/a_{2n}} \leqslant 1 + C_5 \log(ClJ), \end{aligned}$$

by the left inequality in (2.11) and (2.9). Thus

$$H \geqslant C_6 l / \sqrt{\log(ClJ)}.$$

It follows that we obtain (5.14) if we choose l charge enough. Then from (5.14) follows

$$g(u) \leqslant \exp\left(\frac{8lMB}{e}\right), \qquad u \in (0, 1].$$

Next, by elementary calculus, g has at most one local extremum in $[1, \infty)$, and this is a minimum. Thus in any subinterval of $[1, \infty)$, g attains its maximum at the endpoints of that interval. In particular, we must only check that $g((2/B) H \log H)$ is bounded. (Note here that by (5.14), $(2/B) H \log H \geqslant e$). But

$$g\left(\frac{2}{B} H \log H\right) = \exp\left(l \log H \left\{16M - \frac{J}{2}\right\} - \frac{Jl}{2} \log\left[\frac{2}{B} \log H\right]\right) \leqslant 1$$

as $J \geqslant 32M$ and $H \geqslant e^{B/2}$. So we have

$$\Delta(x) \leqslant C_7 \int_{n(x-\tau)/Ba_{2lJn}\sqrt{1-\zeta}}^{\infty} \min\left\{1, \frac{1}{u}\right\}^{lJ/2} du$$

and then (5.1) follows as $J \geqslant 4$. ∎

Proof of (5.1) *for* $x \in (\tau^*, a_n]$. Here

$$\begin{aligned}
\Delta(x) &= \frac{w(x) \int_{\tau^*}^{x} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{w(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds} \\
&\leqslant C_1 \frac{\int_{\tau^*}^{x} \exp(Q(s) - Q(x))\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{(1/n)\sqrt{1-\zeta}} \\
&\leqslant C_2 \frac{n}{\sqrt{1-\zeta}} \left(e^{Q([\tau+x]/2) - Q(x)} \int_{\tau^*}^{[\tau+x]/2} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds \right. \\
&\quad \left. + \int_{[\tau+x]/2}^{x} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds \right) \\
&\leqslant C_3 \left(e^{Q([\tau+x]/2) - Q(x)} \left[1 + \frac{n(\tau^* - \tau)}{a_n\sqrt{1-\zeta}}\right]^{-l} + \left[1 + \frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}\right]^{-l} \right)
\end{aligned} \tag{5.15}$$

by (5.12). Here if $\tau^* > [\tau + x]/2$, the first term in the last two lines can be dropped and we already have the desired estimate. In the contrary case, we must estimate the first term. We note that we can assume that $\tau^* < a_n$, for otherwise the current range of of x is empty. We consider two subcases (recall the definition (5.9) of τ^*):

(I) $\tau^* = \tau + 2(a_n/n)\sqrt{1-\zeta}\, H \log H$

We shall show that

$$\Gamma := \frac{Q(x) - Q([\tau + x]/2)}{l \log(1 + n(x-\tau)/a_n\sqrt{1-\zeta})} \geqslant 1. \tag{5.16}$$

Then the first part of the first term in the right-hand side of (5.15) already gives the desired estimate; the second part of that first term can be bounded above by 1. Now since $tQ'(t)$ is increasing,

$$Q'(t) \geqslant \frac{s}{t} Q'(s) \geqslant \frac{1}{2} Q'(s), \qquad t \geqslant s \geqslant \frac{1}{2}.$$

Hence

$$Q(x) - Q\left(\frac{\tau + x}{2}\right) \geqslant \frac{1}{2} Q'(\tau) \left(\frac{x-\tau}{2}\right).$$

Setting

$$u := \frac{n(x-\tau)}{a_n \sqrt{1-\zeta}},$$

we have

$$\Gamma \geqslant \frac{Q'(\tau)\, a_n \sqrt{1-\zeta}\, u}{4nl \log(1+u)} = \frac{u}{H \log(1+u)}.$$

But

$$u \geqslant \frac{n(\tau^* - \tau)}{a_n \sqrt{1-\zeta}} = 2H \log H.$$

Recall from (5.14) that $H \geqslant \mathrm{e}$. Then since the function $u/\log(1+u)$ is increasing for $u \geqslant 2H \log H \geqslant e$, we obtain

$$\Gamma \geqslant \frac{2H \log H}{H \log(1 + 2H \log H)}.$$

Using the inequality $1 + 2t \log t \leqslant t^2$, $t \geqslant 1$, we have

$$\Gamma \geqslant \frac{2 \log H}{\log H^2} = 1.$$

So we have (5.16) and the result.

(II) $\tau^* = a_{2r}$

In this case, from (2.9),

$$\tau^* - \tau = a_{2r} - a_r \sim \frac{a_r}{T(a_r)} \sim \frac{1}{T(\tau)}.$$

Now if $\tau^* \leqslant x \leqslant \tau(1+1/T(\tau))$, then

$$x - \tau \sim \tau^* - \tau$$

and the second part of the first term in the right-hand side of (5.15) already gives the desired estimate (the first part of the first term can be bounded above by 1). If $x > \tau(1+1/T(\tau))$, then

$$\frac{x}{([x+\tau]/2)} \geqslant 1 + \frac{1}{2T(\tau)+1} \geqslant 1 + \frac{1}{3T(\tau)}$$

for τ close to 1, so from (2.1),

$$\frac{Q(x)}{Q([x+\tau]/2)} \geqslant \left(1 + \frac{1}{3T(\tau)}\right)^{C_2 T([x+\tau]/2)} \geqslant C_3 > 1.$$

(Recall that $[x+\tau]/2 > \tau$). Then

$$e^{Q([\tau+x]/2) - Q(x)} \left[1 + \frac{n(\tau^* - \tau)}{a_n \sqrt{1-\zeta}}\right]^{-l} \leqslant e^{-C_4 Q(x)} \left[1 + \frac{C_5 n}{a_n T(\tau)\sqrt{1-\zeta}}\right]^{-l}.$$

This will admit the desired estimate, namely,

$$C_6 \left[1 + \frac{n(x-\tau)}{a_n \sqrt{1-\zeta}}\right]^{-l}$$

provided

$$e^{C_4 Q(x)/l} \frac{1}{T(\tau)} \geqslant C_7 (x - \tau).$$

But

$$e^{C_4 Q(x)/l} \frac{1}{T(\tau)} \geqslant C_8 \frac{e^{C_4 Q(x)/l}}{T(x)} \geqslant C_9 Q(x) \geqslant C_{10} > C_{10}(x-\tau)$$

by (2.7), (2.12) and the growth (2.4) of Q, so we have the desired estimate. ∎

Proof of (5.1) *for* $x \in (a_n, a_{2lJn}]$. Here, much as in the previous range,

$$\Delta(x) = \frac{w(x) \int_0^x G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{w(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}$$

$$\leqslant C_2 \frac{n}{\sqrt{1-\zeta}} \left(e^{Q([\tau+x]/2) - Q(x)} \int_0^{[\tau+x]/2} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds \right.$$

$$\left. + \int_{[\tau+x]/2}^{x} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds \right)$$

$$\leqslant C_3 \left\{ e^{Q([\tau+x]/2) - Q(x)} + \left[1 + \frac{n(x-\tau)}{a_n \sqrt{1-\zeta}} \right]^{-l} \right\}.$$

We must show that the first term on the last right-hand side admits a bound that is a constant multiple of the second term on the last right-hand side. Let us write $x = a_v$ (so $v \geqslant n$) and $[\tau+x]/2 = a_u$ (so that $u < v$). If first $u \geqslant n/2$, then

$$Q(x) - Q\left(\frac{\tau+x}{2}\right) \geqslant C_4 Q'(a_{n/2})(\tau - x)$$

$$\geqslant C_5 \frac{n}{a_n} T(a_n)^{1/2} (\tau - x) \geqslant C_6 \frac{n(\tau-x)}{a_n \sqrt{1-\zeta}}$$

by (2.6), (2.9). In this case the result follows. If $u < n/2$,

$$Q(x) - Q\left(\frac{\tau+x}{2}\right) \geqslant Q(a_n) - Q(a_{n/2})$$

$$\geqslant C_7 Q(a_n) \geqslant C_8 n T(a_n)^{-1/2} \geqslant C_9 n^{C_{10}}$$

by (2.7), (2.10). Since

$$\left[1 + \frac{n(x-\tau)}{a_n \sqrt{1-\zeta}} \right]^{-l} \geqslant n^{-C_{11}}$$

The result again follows. ∎

Proof of (5.1) *for* $x \in [-a_{2lJn}, 0)$. Here using the evenness of w and (4.1), (5.11) as before gives

$$\begin{aligned}\Delta(x) &= \frac{w(x) \int_x^0 G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{w(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds} \\ &\leqslant C_2 \frac{n}{\sqrt{1-\zeta}} \left(\int_x^0 V_{n,\zeta} \left(\frac{s}{a_{2lJn}} \right)^{lJ} ds \right) \\ &\leqslant C_3 \left[1 + \frac{n\tau}{\sqrt{1-\zeta}} \right]^{-l}.\end{aligned}$$

Here $\tau \sim \tau + |x| = |x - \tau|$ and the result follows. ∎

6. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. Recall that our moduli of continuity are

$$\omega_{r,p}(f, w, t) := \sup_{0<h\leqslant t} \|w \Delta^r_{h\Phi_t(x)}(f, x, [-1, 1])\|_{L_p(|x| \leqslant a_{1/(2t)})} + \inf_{P \in \mathscr{P}_{r-1}} \|(f-P)\, w\|_{L_p(a_{1/(4t)} \leqslant |x| \leqslant 1)}$$

and

$$\bar{\omega}_{r,p}(f, w, t) := \left(\frac{1}{t} \int_0^t \|w \Delta^r_{h\Phi_t(x)}(f, x, [-1, 1])\|^p_{L_p(|x| \leqslant a_{1/(2t)})}\, dh \right)^{1/p} + \inf_{P \in \mathscr{P}_{r-1}} \|(f-P)\, w\|_{L_p(a_{1/(4t)} \leqslant |x| \leqslant 1)}.$$

Of course $\bar{\omega}_{r,p} \leqslant \omega_{r,p}$. We need further moduli of continuity. If I is an interval, and $f: I \to \mathbb{R}$, we define for $t > 0$

$$\Lambda_{r,p}(f, t, I) := \sup_{0<h\leqslant t} \left(\int_I |\Delta^r_h(f, x, I)|^p\, dx \right)^{1/p} \tag{6.1}$$

and its averaged cousin

$$\Omega_{r,p}(f, t, I) := \left(\frac{1}{t} \int_0^t \int_I |\Delta^r_s(f, x, I)|^p\, dx\, ds \right)^{1/p}. \tag{6.2}$$

Note that for some C_1, C_2 depending only on r and p (not on f, I, t),

$$C_1 \leqslant \Lambda_{r,p}(f, t, I)/\Omega_{r,p}(f, t, I) \leqslant C_2. \tag{6.3}$$

See [17, p. 191]. For large enough n, we choose a partition

$$-a_n = \tau_{0n} < \tau_{1n} < \cdots < \tau_{nn} = a_n \tag{6.4}$$

such that if

$$I_{kn} := [\tau_{kn}, \tau_{k+1,n}], \qquad 0 \leqslant k \leqslant n-1, \tag{6.5}$$

then uniformly in k and n,

$$|I_{kn}| \sim \frac{1}{n}\sqrt{1 - \frac{|\tau_{kn}|}{a_{2n}}}. \tag{6.6}$$

($|I|$ denotes the length of the interval I.) We also set $I_{nn} := \varnothing$. There are many ways to do this. For example, one can choose $\tau_{0n} := -a_n$ and for $1 \leqslant k \leqslant n$, determine τ_{kn} by

$$\int_{\tau_{k-1,n}}^{\tau_{kn}} \frac{1}{\sqrt{1-|s|/a_{2n}}}\,ds \Bigg/ \int_{-a_n}^{a_n} \frac{1}{\sqrt{1-|s|/a_{2n}}}\,ds = \frac{1}{n}.$$

Let us set

$$I_n := [-a_n, a_n] = \bigcup_{k=0}^{n-1} I_{kn} \tag{6.7}$$

and ($\chi_{[a,b]}$ denotes the characteristic function of $[a, b]$)

$$\theta_{kn}(x) := \chi_{[\tau_{kn}, a_n]}(x) = \chi_{\bigcup_{i=k}^{n-1} I_{in}}(x) \tag{6.8}$$

We set

$$I_{kn}^* := I_{kn} \cup I_{k+1,n}, \; 0 \leqslant k \leqslant n-1. \tag{6.9}$$

By Whitney's theorem [17, p. 195], we can find for $0 \leqslant k \leqslant n-1$ a polynomial p_k of degree at most r, such that

$$\|f - p_k\|_{L_p(I_{kn}^*)} \leqslant C_2 \Lambda_{r,p}(f, |I_{kn}^*|, I_{kn}^*) \tag{6.10}$$

with $C_2 \neq C_2(f, n, k, I_{kn}^*)$.

Now define an approximating piecewise polynomial/spline by

$$L_n[f](x) := p_0(x)\,\theta_{0n}(x) + \sum_{k=1}^{n-1} (p_k - p_{k-1})(x)\,\theta_{kn}(x). \tag{6.11}$$

We first show that $L_n[f]$ is a good approximation to f:

LEMMA 6.1. *Let $\Psi_n : [-a_n, a_n] \to \mathbb{R}$ be such that uniformly in n,*

$$\Psi_n(x) \sim \sqrt{1 - \frac{|x|}{a_{2n}}}, \qquad x \in [-a_n, a_n]. \tag{6.12}$$

Then

$$\begin{aligned} &\|(f - L_n[f])\, w\|_{L_p[-1,1]} \\ &\quad \leqslant C_1 \left\{ \left[n \int_0^{C_2/n} \|w \Delta^r_{h\Psi_n(x)}(f, x, [-1, 1])\|^p_{L_p[-a_n, a_n]}\, dh \right]^{1/p} \right. \\ &\qquad \left. + \|fw\|_{L_p(a_n \leqslant |x| \leqslant 1)} \right\}. \end{aligned} \tag{6.13}$$

Here $C_j \neq C_j(f, n)$, $j = 1, 2$. For $p = \infty$, we replace the pth root and integral by $\sup_{0 < h \leqslant C_2 n}$. Moreover, the constants are independent of $\{\Psi_n\}$, depending only on the constants in $\sim$ in (6.12).

Proof. We first deal with $p < \infty$. Now

$$\|(f - L_n[f])\, w\|^p_{L_p[-1,1]} = \sum_{j=0}^{n-1} \Delta_{jn} + \|fw\|^p_{L_p(a_n \leqslant |x| \leqslant 1)}, \tag{6.14}$$

where

$$\Delta_{jn} := \int_{I_{jn}} |f - L_n[f]|^p\, w^p. \tag{6.15}$$

Note that in $(\tau_{jn}, \tau_{j+1,n})$, $L_n[f] = p_j$, so that

$$\begin{aligned} \Delta_{jn} &= \int_{I_{jn}} |f - p_j|^p\, w^p \\ &\leqslant \|w\|^p_{L_\infty(I_{jn})}\, C_2^p\, \Lambda^p_{r,p}(f, |I^*_{jn}|, I^*_{jn}) \\ &\leqslant \|w\|^p_{L_\infty(I^*_{jn})}\, \|w^{-1}\|^p_{L_\infty(I^*_{jn})} \frac{C_3}{|I^*_{jn}|} \int_0^{|I^*_{jn}|} \int_{I^*_{jn}} |w \Delta^r_s(f, x, I^*_{jn})|^p\, dx\, ds, \end{aligned} \tag{6.16}$$

by (6.2), (6.3). Now from (3.4) of Lemma 3.2(a),

$$\|w\|^p_{L_\infty(I^*_{jn})}\, \|w^{-1}\|^p_{L_\infty(I^*_{jn})} \sim 1 \tag{6.17}$$

uniformly in j and n. Moreover, uniformly in j, n, and $x \in I^*_{jn}$,

$$|I^*_{jn}| \sim \frac{1}{n} \sqrt{1 - \frac{|x|}{a_{2n}}} \sim \frac{1}{n} \Psi_n(x).$$

Then we can continue (6.16) as

$$\Delta_{jn} \leqslant \frac{C_4}{|I_{jn}^*|} \int_{I_{jn}^*} \int_0^{|I_{jn}^*|} |w\Delta_s^r(f, x, I_{jn}^*)|^p \, ds \, dx$$

$$= \frac{C_4}{|I_{jn}^*|} \int_{I_{jn}^*} \Psi_n(x) \int_0^{|I_{jn}^*|/\Psi_n(x)} |w\Delta_{t\Psi_n(x)}^r(f, x, I_{jn}^*)|^p \, dt \, dx$$

$$\leqslant C_5 n \int_0^{C_6/n} \int_{I_{jn}^*} |w\Delta_{t\Psi_n(x)}^r(f, x, I_{jn}^*)|^p \, dx \, dt. \tag{6.18}$$

Adding over j gives

$$\sum_{j=0}^{n-1} \Delta_{jn} \leqslant C_5 n \int_0^{C_6/n} \int_{I_n} |w\Delta_{t\Psi_n(x)}^r(f, x, [-1, 1])|^p \, dx \, dt. \tag{6.19}$$

This and (6.14) give the result. Note that we have effectively also shown that

$$\sum_{j=0}^{n-1} \Omega_{r,p}(f, |I_{jn}^*|, I_{jn}^*)^p \, w^p(\tau_{jn})$$

$$\leqslant C_5 n \int_0^{C_6/n} \int_{I_n} |w\Delta_{t\Psi_n(x)}^r(f, x, [-1, 1])|^p \, dx \, dt. \tag{6.20}$$

For $p = \infty$, the proof is similar, but easier: We see that

$$\|(f - L_n[f]) \, w\|_{L_\infty(-1,1)}^p$$

$$\leqslant \max\Big\{ \max_{0 \leqslant j \leqslant n-1} \|(f - p_j) \, w\|_{L_\infty(I_{jn})}, \|fw\|_{L_\infty(a_n \leqslant |x| \leqslant 1)} \Big\}.$$

The rest of the proof is as before. ∎

Now we can define our polynomial approximation to f:

$$P_n[f] := p_0(x) \, R_{n, \tau_{on}}(x) + \sum_{k=1}^{n-1} (p_k - p_{k-1})(x) \, R_{n, \tau_{kn}}(x). \tag{6.21}$$

Note that this has been formed from $L_n[f]$ by replacing the characteristic function $\theta_{kn}(x) = \chi_{[\tau_{kn}, a_n]}(x)$ with its polynomial approximation $R_{n, \tau_{kn}}(x)$ formed in the previous section.

LEMMA 6.2. *Let $\{\Psi_n\}$ be as in the previous lemma. Then*

$$\|(L_n[f]-P_n[f])\,w\|_{L_p(-1,1)}$$
$$\leqslant C_1\left\{\left[n\int_0^{C_2/n}\|w\Delta^r_{h\Psi_n(x)}(f,x,[-1,1])\|^p_{L_p[-a_n,a_n]}\,dh\right]^{1/p}+\|fw\|_{L_p(I^*_{0n})}\right\}. \tag{6.22}$$

For $p=\infty$, we replace the pth root and integral by $\sup_{0<h\leqslant C_2/n}$.

Proof. We see that if we define $p_{-1}(x)\equiv 0$,

$$(L_n[f]-P_n[f])(x)$$
$$=\sum_{k=0}^{n-1}(p_k-p_{k-1})(x)(\theta_{kn}(x)-R_{n,\tau_{kn}}(x)). \tag{6.23}$$

We shall make substantial use of the following inequality: Let S be a polynomial of degree at most r and $[a, b]$ be a real interval. Then for all $x\in[-1, 1]$,

$$|S(x)|\leqslant C(b-a)^{-1/p}\left(1+\frac{\min\{|x-a|,|x-b|\}}{b-a}\right)^r\|S\|_{L_p[a,b]}. \tag{6.24}$$

Here $C\neq C(a, b, x, S)$ but $C=C(p, r)$. This follows from standard Nikolskii inequalities and the Bernstein–Walsh inequality. See for example [17, p. 193]. Hence for $x\in[-1, 1]$, and $1\leqslant k\leqslant n-1$,

$$|p_k-p_{k-1}|\,(x)\leqslant C\,|I_{kn}|^{-1/p}\left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^r\|p_k-p_{k-1}\|_{L_p(I_{kn})}.$$

This is still true for $k=0$ if we recall that $p_{-1}\equiv 0$. Now for $1\leqslant k\leqslant n-1$, (6.10) gives

$$\|p_k-p_{k-1}\|_{L_p(I_{kn})}\leqslant C_1\sum_{i=k-1}^{k}\Lambda_{r,p}(f,|I^*_{in}|,I^*_{in}),$$

where $C_1\neq C_1(f, k, n)$. This remains true for $k=0$ if we set

$$\Omega_{r,p}(f,|I^*_{-1,n}|,I^*_{-1,n}):=\Lambda_{r,p}(f,|I^*_{-1,n}|,I^*_{-1,n}):=\|f\|_{L_p(I^*_{on})}.$$

Since (see (3.5), (6.6)) uniformly in k, n, and $x \in [-1, 1]$,

$$1 + \frac{|x - \tau_{kn}|}{|I_{kn}|} \sim 1 + \frac{|x - \tau_{k-1,n}|}{|I_{k-1,n}|}$$

we obtain from Theorem 5.1, uniformly for $0 \leqslant k \leqslant n-1$ and $x \in [-1, 1]$,

$$\begin{aligned} &|(p_k - p_{k-1})(x)(\theta_{kn}(x) - R_{n,\tau_{kn}}(x))| \frac{w(x)}{w(\tau_{kn})} \\ &\quad \leqslant C_2 \sum_{i=k-1}^{k} |I_{in}|^{-1/p} \left(1 + \frac{|x - \tau_{in}|}{|I_{in}|}\right)^{r-l} \Omega_{r,p}(f, |I_{in}^*|, I_{in}^*). \end{aligned} \tag{6.25}$$

Here and in the sequel, we set $|I_{-1,n}| := |I_{0,n}|$ and $\tau_{-1,n} := \tau_{0,n}$. We consider three different ranges of p:

(I) $0 < p < 1$.

Here from (6.23) and then (6.25)

$$\begin{aligned} &\int_{-1}^{1} (|L_n[f] - P_n[f]|\, w)^p \\ &\quad \leqslant \sum_{k=0}^{n-1} \int_{-1}^{1} (|p_k - p_{k-1}|\, |\theta_{kn} - R_{n,\tau_{kn}}|\, w)^p \\ &\quad \leqslant \sum_{k=-1}^{n-1} |I_{kn}|^{-1}\, \Omega_{r,p}^p(f, |I_{kn}^*|, I_{kn}^*)\, w^p(\tau_{kn}) \int_{-1}^{1} \left(1 + \frac{|x - \tau_{kn}|}{|I_{kn}|}\right)^{(r-l)p} dx. \end{aligned} \tag{6.26}$$

Here if $(r-l)\,p < -1$,

$$|I_{kn}|^{-1} \int_{-1}^{1} \left(1 + \frac{|x - \tau_{kn}|}{|I_{kn}|}\right)^{(r-l)p} dx \leqslant \int_{-\infty}^{\infty} (1 + |u|)^{(r-l)p}\, du =: C_3 < \infty.$$

So

$$\int_{-1}^{1} (|L_n[f] - P_n[f]|\, w)^p \leqslant C_4 \sum_{k=-1}^{n-1} \Omega_{r,p}^p(f, |I_{kn}^*|, I_{kn}^*)\, w^p(\tau_{kn}).$$

This is the same as our sum in (6.20)), except for the term for $k = -1$. So the estimate (6.20) gives (6.22), keeping in mind our choice of $\Lambda_{r,p}(f, |I_{-1,n}^*|, I_{-1,n}^*)$.

(II) $1 \leqslant p < \infty$.

From (6.23), (6.25) and then Hölder's inequality,

$$\{|L_n[f]-P_n[f]|\,(x)\,w(x)\}^p$$
$$\leqslant C\left\{\sum_{k=-1}^{n-1} |I_{kn}|^{-1/p}\left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^{r-l} \Omega_{r,p}(f, |I^*_{kn}|, I^*_{kn})\, w(\tau_{kn})\right\}^p$$
$$\leqslant C \sum_{k=-1}^{n} |I_{kn}|^{-1}\left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^{(r-l)p/2} \Omega^p_{r,p}(f, |I^*_{kn}|, I^*_{kn})\, w^p(\tau_{kn}) \cdot S_n(x)^{p/q}, \tag{6.27}$$

where $q = p/(p-1)$ and

$$S_n(x) := \sum_{k=1}^{n}\left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^{(r-l)q/2}.$$

We shall show that if $(r-l)\,q/2 < -1$, then

$$\sup_{n\geqslant 1}\ \sup_{x\in[-1,1]} S_n(x) \leqslant C_1 < \infty. \tag{6.28}$$

Note that $S_n(x)$ is a decreasing function of x for $x \geqslant a_n = \tau_{nn}$, so it suffices to consider $x \in [0, a_n]$. Recall that

$$|I_{kn}| \sim |I_{k+1,n}| \sim \frac{1}{n}\sqrt{1-\frac{|\tau_{kn}|}{a_{2n}}}.$$

It is then not difficult to see that

$$S_n(x) \leqslant C_2 n \int_{-a_n}^{a_n}\left(1+n\frac{|x-u|}{\sqrt{1-|u|/a_{2n}}}\right)^{(r-l)q/2} \frac{du}{\sqrt{1-|u|/a_{2n}}}$$
$$\leqslant C_3 n \int_{-1}^{1}\left(1+n\frac{|\bar{x}-s|}{\sqrt{1-s}}\right)^{(r-l)q/2} \frac{ds}{\sqrt{1-s}},$$

where $\bar{x} := x/a_{2n}$, so that

$$1-\bar{x} \geqslant 1-a_n/a_{2n} \geqslant C_4\, T(a_n)^{-1} \geqslant C_5 n^{-2}.$$

We make the substitution $(1-s)=(1-\bar{x})\,w$ to obtain

$$S_n(x)\leqslant C_3 n\sqrt{1-\bar{x}}\int_0^{2/(1-\bar{x})}\left(1+n\sqrt{1-\bar{x}}\,\frac{|w-1|}{\sqrt{w}}\right)^{(r-l)q/2}\frac{dw}{\sqrt{w}}$$

$$\leqslant C_4 n\sqrt{1-\bar{x}}\left\{\int_0^{1/2}\left[1+\frac{n\sqrt{1-\bar{x}}}{\sqrt{w}}\right]^{(r-l)\,q/2}\frac{dw}{\sqrt{w}}\right.$$

$$+\int_{1/2}^{3/2}\left[1+n\sqrt{1-\bar{x}}\,|w-1|\right]^{(r-l)\,q/2}dw$$

$$\left.+\int_{3/2}^{2/(1-\bar{x})}\left[1+n\sqrt{(1-\bar{x})\,w}\right]^{(r-l)\,q/2}\frac{dw}{\sqrt{w}}\right\}.$$

(We can omit the third integral if $2/(1-\bar{x})\leqslant 3/2$.) We now make the substitutions $w=n^2(1-\bar{x})\,v$ in the first integral, $v=n\sqrt{1-\bar{x}}\,(w-1)$ in the second integral, and $v=n^2(1-\bar{x})\,w$ in the third integral. It is then not difficult to see that the resulting terms are bounded independent of n and x if l is large enough. So we have (6.28). Then using this, integrating (6.27) (we can assume that $(r-l)\,p/2<-1$) and using (6.20) gives the result.

(III) $p=\infty$.

Now by (6.23), (6.25)

$$|L_n[f]-P_n[f]|\,(x)\,w(x)$$

$$\leqslant C\sum_{k=0}^{n-1}|p_k-p_{k-1}|\,(x)\,|\theta_{kn}-R_{n,\,\tau_{kn}}|\,(x)\,w(x)$$

$$\leqslant C\max_{-1\leqslant k\leqslant n-1}\Omega_{r,p}(f,|I^*_{kn}|,I^*_{kn})\,w(\tau_{kn})\cdot\sum_{k=0}^{n-1}\left(1+\frac{|x-\tau_{kn}|}{|I_{kn}}\right)^{(r-l)}.$$

As before, the sum is bounded if l is large enough. Then we can continue this as

$$\leqslant C_1\{\sup_{0\leqslant k\leqslant n-1}\ \sup_{0<h\leqslant|I^*_{kn}|}\|\Delta^r_h(f,x,I^*_{kn})\,w\|_{L_\infty(I^*_{kn})}+\|fw\|_{L_\infty(I^*_{0n})}\}$$

$$\leqslant C_2\{\sup_{0\leqslant k\leqslant n-1}\ \sup_{0<h\leqslant C/n}\|\Delta^r_{h\Psi_n(x)}(f,x,I^*_{kn})\,w\|_{L_\infty(I^*_{kn})}+\|fw\|_{L_\infty(I^*_{0n})}\}$$

$$\leqslant C_3\{\sup_{0<h\leqslant C/n}\|\Delta^r_{h\Psi_n(x)}(f,x,[-1,1])\,w\|_{L_\infty(-a_n,a_n)}+\|fw\|_{L_\infty(I^*_{0n})}\}.\quad\blacksquare$$

We can now turn to the

Proof of Theorem 1.2. We do this for $p<\infty$; the case $p=\infty$ is similar, but much easier. Now recall that $R_{n,\tau}$ has degree at most $2lJn$, where J is as in Theorem 5.1. So $P_n[f]$ has degree at most $2lJn+r$. So, if $M := 3lJ$, we have for large n

$$\begin{aligned} E_{Mn}[f]_{w,p} &\leqslant \|(f-P_n[f])\,w\|_{L_p(-1,1)} \\ &\leqslant C\{\|(f-L_n[f])\,w\|_{L_p(-1,1)} + \|(L_n[f]-P_n[f])\,w\|_{L_p(-1,1)}\} \\ &\leqslant C_1 \left\{\left[n\int_0^{C_2/n} \|w\Delta^r_{h\Psi_n(x)}(f,x,[-1,1])\|^p_{L_p(-a_n,a_n)}\,dh\right]^{1/p}\right. \\ &\quad \left. + \|fw\|_{L_p(a_n(1-C_2[nT(a_n)^{1/2}]^{-1})\leqslant |x|\leqslant 1)}\right\}. \end{aligned} \tag{6.29}$$

Here we have used Lemmas 6.1 and 6.2, and also (6.6), which implies that

$$|I^*_{0n}| \sim \frac{1}{n}\sqrt{1-\frac{a_n}{a_{2n}}} \sim \frac{1}{n}\,T(a_n)^{-1/2}.$$

Next for

$$Mn \leqslant j < M(n+1) \tag{6.30}$$

we write

$$n = \kappa j,$$

where $\kappa = \kappa(j, n)$. Note that

$$\kappa = \frac{n}{j} \to \frac{1}{M}, \qquad j \to \infty. \tag{6.31}$$

Let

$$t := t(j) = \frac{M}{2j}.$$

From (6.30) and (6.31), we have for $n \geqslant 2$

$$n \leqslant \frac{j}{M} = \frac{1}{2t};\ n \geqslant \frac{2}{3}\,\frac{j}{M} = \frac{1}{3t}.$$

We claim that for large enough j,

$$a_n(1-C_2[nT(a_n)^{1/2}]^{-1}) \geqslant a_{1/(4t)}.$$

To see this, note from (2.12) that

$$[nT(a_n)^{1/2}]^{-1} = o(T(a_n)^{-1})$$

so that by (2.9)

$$a_n(1 - C_2[nT(a_n)^{1/2}]^{-1}) \geqslant a_n\left(1 - o\left(\frac{1}{T(a_n)}\right)\right) \geqslant a_{2n/3}$$

$$= a_{(1+o(1))\,2j/3M} = a_{(1+o(1))/(3t)} \geqslant a_{1/(4t)}$$

for large enough j. Then from (6.29),

$$\begin{aligned} E_j[f]_{w,p} &\leqslant E_{Mn}[f]_{w,p} \\ &\leqslant C_1\left\{\left[\frac{1}{2t}\int_0^{3C_2 t} \|w\Delta^r_{h\Psi_n(x)}(f, x, [-1, 1])\|^p_{L_p(-a_{1/(2t)},\, a_{1/(2t)})}\, dh\right]^{1/p}\right. \\ &\quad \left. + \|fw\|_{L_p(a_{1/(4t)} \leqslant |x| \leqslant 1)}\right\}. \end{aligned} \tag{6.32}$$

Now we choose

$$\Psi_n := (3C_2)^{-1}\,\Phi_t.$$

We must show that (6.12) holds with constants independent of x, j and n, that is,

$$(3C_2)^{-1}\,\Phi_t(x) \sim \sqrt{1 - \frac{|x|}{a_{2n}}}, \qquad |x| \leqslant a_n.$$

But for this range of x, (2.9) shows that

$$\sqrt{1 - \frac{|x|}{a_{2n}}} \sim \sqrt{1 - \frac{|x|}{a_{2n}}} + T(a_{2n})^{-1/2} = \Phi_{1/2n}(x) \sim \Phi_t(x)$$

by Lemma 3.1(b). Setting $h_1 := h/(3C_2)$ so that $h\Psi_n = h_1\Phi_t$ we can rewrite (6.32) as

$$\begin{aligned} E_j[f]_{w,p} &\leqslant C_1\left\{\left[\frac{3C_2}{2t}\int_0^{t} \|w\Delta^r_{h_1\Phi_t(x)}(f, x, [-1, 1])\|^p_{L_p(-a_{1/(2t)},\, a_{1/(2t)})}\, dh_1\right]^{1/p}\right. \\ &\quad \left. + \|fw\|_{L_p(a_{1/(4t)} \leqslant |x| \leqslant 1)}\right\}. \end{aligned}$$

Replacing f by $f-P$ for suitable $P \in \mathscr{P}_{r-1}$, and using $\Delta^r_{h_1 \Phi_t(x)}(P, x, [-1, 1]) \equiv 0$, we obtain

$$\begin{aligned} E_j[f]_{w,p} &= E_j[f-P]_{w,p} \\ &\leqslant C_3 \left\{ \left[\frac{1}{t} \int_0^t \| w \Delta^r_{h_1 \Phi_t(x)}(f, x, [-1, 1]) \|^p_{L_p(-a_{1/(2t)}, a_{1/(2t)})} \, dh_1 \right]^{1/p} \right. \\ &\quad \left. + \inf_{P \in \mathscr{P}_{r-1}} \| (f-P) w \|_{L_p(a_{1/(4t)} \leqslant |x| \leqslant 1)} \right\} \\ &= C_3 \bar{\omega}_{r,p}(f, w, t) = C_3 \bar{\omega}_{r,p}\left(f, w, \frac{M}{2j}\right). \quad \blacksquare \end{aligned}$$

For future use, we record a slight generalization of Theorem 1.2:

THEOREM 6.3. *For* $j \geqslant C_3$,

$$E_j[f]_{w,p} \leqslant C_1 \inf_{\rho \in [3/4, 1]} \bar{\omega}_{r,p}\left(f, w, \frac{C_2 \rho}{j}\right), \tag{6.33}$$

where $C_k \neq C_k(j, \rho, f)$, $k = 1, 2, 3$.

Proof. The only difference to the above proof is that we choose $t := M\rho/2j$. Then uniformly for $\rho \in [\frac{3}{4}, 1]$,

$$nt = \frac{nM\rho}{2j} \to \frac{\rho}{2}, \qquad j \to \infty$$

and as $\rho/2 \geqslant 3/8 > 1/3$, we have for $j \geqslant j_0 \neq j_0(\rho, f, t)$

$$\frac{1}{2t} \geqslant n \geqslant \frac{1}{3t}.$$

The previous considerations then remain the same, as does our choice of Ψ_n, the point being that (6.12) holds uniformly in ρ. $\blacksquare$

7. THE PROOF OF THEOREM 1.3

We begin with a technical lemma:

LEMMA 7.1. (a) *For* $0 < s < t \leqslant C$,

$$T(a_{1/t}) \left(1 - \frac{a_{1/t}}{a_{1/s}}\right) \leqslant C_1 \log\left(C_2 \frac{t}{s}\right). \tag{7.1}$$

(b) *For* $0<s<t\leqslant C$,

$$\sup_{x\in[-1,1]}\frac{\Phi_s(x)}{\Phi_t(x)}\leqslant C_2\sqrt{\log\left(2+\frac{t}{s}\right)}. \tag{7.2}$$

Hence, given $\gamma>0$,

$$\sup_{x\in[-1,1]}\left(\frac{s}{t}\right)^{\gamma}\frac{\Phi_s(x)}{\Phi_t(x)}\leqslant C_3. \tag{7.3}$$

Proof. (a) Using the inequality $1-u\leqslant\log(1/u)$, $u\in(0,1]$, we obtain

$$1-\frac{a_{1/t}}{a_{1/s}}\leqslant\log\frac{a_{1/s}}{a_{1/t}}\leqslant C_4\frac{\log(C\,t/s)}{T(a_{1/t})},$$

by (2.11).

(b) Now if $x\geqslant 0$,

$$\left|1-\frac{x}{a_{1/s}}\right|\leqslant\left|1-\frac{x}{a_{1/t}}\right|+\frac{x}{a_{1/t}}\left|1-\frac{a_{1/t}}{a_{1/s}}\right|$$
$$\leqslant\left|1-\frac{x}{a_{1/t}}\right|+C_5T(a_{1/t})^{-1}\log\left(C\frac{t}{s}\right)$$

by (a) provided $t\leqslant C$, say. We deduce that

$$\left|1-\frac{x}{a_{1/s}}\right|^{1/2}\leqslant C_6\Phi_t(x)\sqrt{\log\left(2+\frac{t}{s}\right)},$$

and since also $T(a_{1/s})^{-1/2}\leqslant C_7T(a_{1/t})^{-1/2}$, we obtain (7.2). Then (7.3) also follows. ∎

We turn to the proof of Theorem 1.3. We provide full proofs only where the details are significantly different and otherwise refer back for proofs. We begin with an analogue of Lemma 6.1 for $L_n[f]$ of (6.11).

LEMMA 7.2.

$$\|(f-L_n[f])\,w\|_{L_p[-1,1]}$$
$$\leqslant C_1\Big\{\sup_{\substack{0<h\leqslant 1/(3n)\\ 0<\tau\leqslant L}}\|w\Delta^r_{\tau h\Phi_h(x)}(f,x,[-1,1])\|_{L_p[-a_{1/(2h)},a_{1/(2h)}]}$$
$$+\|fw\|_{L_p(a_n\leqslant|x|\leqslant 1)}\Big\}. \tag{7.4}$$

Here L is independent of f, n.

Proof. We do this for $0<p<\infty$; the case $p=\infty$ is simpler. Recall that the crux of Lemma 6.1 is estimation of

$$\begin{aligned}\Delta_{jn} &:= \int_{I_{jn}} (|f-p_j|\, w)^p \\ &\leqslant C_1 \Omega_{r,p}(f, |I^*_{jn}|, I^*_{jn})^p\, w^p(\tau_{jn}) \\ &\leqslant \frac{C_2}{|I^*_{jn}|} \int_{I^*_{jn}} \int_0^{|I^*_{jn}|} |w\Delta^r_s(f, x, I^*_{jn})|^p\, ds\, dx. \end{aligned} \tag{7.5}$$

(See (6.16).) We now choose $L>0$ such that

$$\sup_{x\in(-1,1)} \frac{(h/L)\, \Phi_{h/L}(x)}{h\Phi_h(x)} \leqslant \frac{1}{2}, \qquad 0<h\leqslant 1. \tag{7.6}$$

This is possible by (7.2). Now we choose

$$\delta_{n,k}(x) := L^{1-k}(3n)^{-1}\, \Phi_{L^{1-k}(3n)^{-1}}(x), \qquad k\geqslant 1.$$

Note that by (7.6),

$$\sup_{x\in(-1,1)} \frac{\delta_{n,k+1}(x)}{\delta_{n,k}(x)} \leqslant \frac{1}{2}. \tag{7.7}$$

In view of (6.6), (3.6), and (3.7), we may assume that L is so large that uniformly in $n, j, x\in I^*_{jn}$,

$$|I^*_{jn}| \leqslant \frac{L}{3n}\, \Phi_{1/3n}(x) = L\delta_{n,1}(x);$$

and

$$|I^*_{jn}| \sim \delta_{n,1}(x).$$

Then from (7.5),

$$\begin{aligned}\Delta_{jn} &\leqslant C_3 \int_{I^*_{jn}} \int_0^{L\delta_{n,1}(x)} \frac{1}{\delta_{n,1}(x)} |w\Delta^r_s(f, x, I^*_{jn})|^p\, ds\, dx \\ &= C_3 \int_{I^*_{jn}} \sum_{k=1}^{\infty} \int_{L\delta_{n,k+1}(x)}^{L\delta_{n,k}(x)} \frac{1}{\delta_{n,1}(x)} |w\Delta^r_s(f, x, I^*_{jn})|^p\, ds\, dx \\ &= C_3 \int_{I^*_{jn}} \sum_{k=1}^{\infty} \frac{\delta_{n,k}(x)}{\delta_{n,1}(x)} \int_{L\delta_{n,k+1}(x)/\delta_{n,k}(x)}^{L} |w\Delta^r_{\tau\delta_{n,k}(x)}(f, x, I^*_{jn})|^p\, d\tau\, dx \\ &\leqslant C_4 \int_{I^*_{jn}} \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} \int_0^L |w\Delta^r_{\tau\delta_{n,k}(x)}(f, x, I^*_{jn})|^p\, d\tau\, dx. \end{aligned}$$

Then as also $n \leqslant 1/(2h)$ for $0 < h \leqslant 1/(3n)$,

$$\sum_{j=0}^{n-1} \Delta_{jn} \leqslant C_5 \int_{-a_n}^{a_n} \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^{k-1} \int_0^L |w\Delta^r_{\tau\delta_{n,k}(x)}(f, x, (-1, 1))|^p \, d\tau \, dx$$

$$\leqslant 2C_5 \sup_{\substack{0 < h \leqslant 1/(3n) \\ 0 < \tau \leqslant L}} \int_{-a_{1/(2h)}}^{a_{1/(2h)}} |w\Delta^r_{\tau h \Phi_h(x)}(f, x, (-1, 1))|^p \, dx. \qquad (7.8)$$

The rest of the proof is as before. ∎

We turn to the

Proof of Theorem 1.3. The method of proof of Lemma 6.2 gives at least for $p < \infty$,

$$\|(L_n[f] - P_n[f])\, w\|^p_{L_p[-1,1]}$$

$$\leqslant C_1 \left\{ \sup_{\substack{0 < h \leqslant 1/(3n) \\ 0 < \tau \leqslant L}} \int_{-a_{1/(2h)}}^{a_{1/(2h)}} |w\Delta^r_{\tau h \Phi_h(x)}(f, x, (-1,1))|^p \, dx + \|fw\|_{L_p(I^*_{0n})} \right\}.$$

(We substitute for (6.20) the appropriate estimate (7.8) in the relevant places.) The rest of the estimation is almost the same as in the proof of Theorem 1.2. We can still choose $t := M/(2j)$ and still have $1/(3n) \leqslant t$. ∎

Finally, we briefly show that under some additional conditions on Q, we can use the simpler modulus

$$\omega^{\#}_{r,p}(f, w, t) := \sup_{0 < h \leqslant t} \|w\Delta^r_{Lh\Phi_h(x)}(f, x, (-1, 1))\|_{L_p(-a_{1/(2h)}, a_{1/(2h)})}$$

$$+ \inf_{P \in \mathscr{P}_{r-1}} \|(f - P)\, w\|_{L_p(a_{1/(4t)} \leqslant |x| \leqslant 1)}, \qquad (7.9)$$

with L fixed as above. We do this for $p < \infty$; $p = \infty$ is easier. We shall assume in addition to $w \in \mathscr{E}$ that Q'' exists and is non-negative in $(0, 1)$, and that

$$\frac{Q''(x)}{Q'(x)} \sim \frac{Q'(x)}{Q(x)}, \qquad x \in (0, 1) \qquad (7.10)$$

and

$$|T'(x)| \leqslant C_1 T^2(x), \qquad x \in (C, 1). \qquad (7.11)$$

Using (7.10) and the method of proof of Lemma 3.2 in [10, p. 24] we obtain

$$\frac{a'_u}{a_u} \sim \frac{1}{uT(a_u)}, \qquad u \in (0, \infty). \tag{7.12}$$

(Note that T has a different meaning in [10], but has the same rate of growth as the T here, because of (7.10).) Moreover, using (7.11) and (7.12) it is not difficult to see that

$$\left| \frac{d}{dt} (tT(a_{1/t})^{-1/2}) \right| \leqslant C_2 T(a_{1/t})^{-1/2}, \qquad t \in (0, C)$$

and hence also

$$\left| \frac{d}{dt} (t\Phi_t(x)) \right| \leqslant C_3 \Phi_t(x) \tag{7.13}$$

for

$$0 < t \leqslant C_4; \qquad \left| 1 - \frac{|x|}{a_{1/t}} \right| \geqslant \frac{\varepsilon}{T(a_{1/t})}. \tag{7.14}$$

Here ε is any fixed positive number. We now estimate Δ_{jn} a little differently from the way we proceeded after (7.5). Let us make the substitution $s = Lt\Phi_t(x)$ in the right-hand side of (7.5), keep our choice of L there, and recall that

$$|I^*_{jn}| \leqslant \frac{L}{3n} \Phi_{1/3n}(x), \qquad x \in I^*_{jn}$$

to deduce that

$$\begin{aligned} \Delta_{jn} &\leqslant C_5 \int_{I^*_{jn}} \int_0^{1/(3n)} \frac{1}{(1/(3n))\, \Phi_{1/(3n)}(x)} |w \Delta^r_{Lt\Phi_t(x)}(f, x, I^*_{jn})|^p \left| \frac{d}{dt} [t\Phi_t(x)] \right| dt\, dx \\ &\leqslant C_6 n \int_{I^*_{jn}} \int_0^{1/(3n)} \sqrt{\log\left(2 + \frac{1}{3nt}\right)} |w \Delta^r_{Lt\Phi_t(x)}(f, x, I^*_{jn})|^p \, dt\, dx \end{aligned}$$

by first (7.13) and then (7.2). In applying (7.13) we must ensure that the range conditions in (7.14) must hold for $x \in I^*_{jn}$ and $t \leqslant 1/3n$. In fact if $|x| \leqslant a_n$, then for $t \leqslant 1/3n$,

$$\left| 1 - \frac{|x|}{a_{1/t}} \right| \geqslant 1 - \frac{a_n}{a_{3n}} \geqslant C_7 T(a_n)^{-1} \geqslant C_8 T(a_{1/t})^{-1}.$$

Thus

$$\sum_{j=0}^{n-1} \Delta_{jn}$$

$$\leqslant C_9 n \int_{-a_n}^{a_n} \int_0^{1/(3n)} \sqrt{\log\left(2+\frac{1}{3nt}\right)}\, |w\Delta^r_{Lt\Phi_t(x)}(f, x, (-1, 1))|^p \, dt\, dx$$

$$\leqslant C_{10} \sup_{0<h\leqslant 1/(3n)} \int_{-a_{1/(2h)}}^{a_{1/(2h)}} |w\Delta^r_{Lh\Phi_h(x)}(f, x, (-1, 1))|^p \, dx \int_0^1 \sqrt{\log\left(2+\frac{1}{s}\right)}\, ds.$$

So under $\omega \in \mathscr{E}$, and the additional conditions (7.10), (7.11) on Q, we obtain

$$E_n[f]_{w,p} \leqslant C_{11} \omega^{\#}_{r,p}\left(f, w, \frac{1}{n}\right). \tag{7.15}$$

We note finally that the additional conditions (7.10) and (7.11) are certainly satisfied for $w_{k,\alpha}$ of (1.5).

ACKNOWLEDGMENTS

This paper was begun in 1992 as a joint project with Vili Totik. Because the methods now used are different from those originally proposed, Vili Totik felt that he could not be a coauthor. Nevertheless, he contributed substantially in terms of discussions, scope of the project, and so on. Many of the ideas of proof of this paper arose out of earlier work with Zeev Ditzian on Freud weights.

REFERENCES

1. J. Clunie and T. Kövari, On integral functions having prescribed asymptotic growth, II, *Canad. J. Math.* **20** (1968), 7–20.
2. S. B. Damelin and D. S. Lubinsky, Jackson theorems for Erdös weights in L_p $(0<p\leqslant\infty)$, to appear in *J. Approx. Theory.*
3. R. A. DeVore and V. A. Popov, Interpolation of Besov spaces, *Trans. Amer. Math. Soc.* **305** (1988), 397–414.
4. R. A. DeVore, D. Leviatan, and X. M. Yu, Polynomial approximation in L_p $(0<p<1)$, *Constr. Approx.* **8** (1992), 187–201.
5. Z. Ditzian and D. S. Lubinsky, Jackson and smoothness theores for Freud weights in L_p, $0<p\leqslant\infty$, *Constr. Approx.* **13** (1997), 99–152.
6. Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer Series in Computational Mathematics, Vol. 9, Springer-Verlag, Berlin, 1987.
7. G. Freud, "Orthogonal Polynomials," Pergamon/Akademiai Kiado, Budapest, 1971.

8. M. von Golitschek, G. G. Lorentz, and Y. Makovoz, Asymptotics of weighted polynomials, *in* "Progress in Approximation Theory" (A. A. Gonchar and E. B. Saff, Eds.), Springer Series in Computational Mathematics, Vol. 19, pp. 431–451, Springer-Verlag, New York, 1992.
9. A. L. Levin and D. S. Lubinsky, Christoffel functions, orthogonal polynomials, and Nevai's conjecture for Freud weights, *Constr. Approx.* **8** (1992), 463–535.
10. A. L. Levin and D. S. Lubinsky, Christoffel functions and orthogonal polynomials for exponential weights on (−1, 1), *Mem. Amer. Soc.* **535** (1994), 111.
11. D. S. Lubinsky and E. B. Saff, Markov–Bernstein and Nikolskii inequalities, and Christoffel functions for exponential weights on (−1, 1), *SIAM J. Math. Anal.* **24** (1993), 528–556.
12. H. N. Mhaskar and E. B. Saff, Extremal problems for polynomials with exponential weights, *Trans. Amer. Math. Soc.* **285** (1984), 203–234.
13. H. N. Mhaskar and E. B. Saff, Where does the sup-norm of a weighted polynomial live? *Constr. Approx.* **1** (1985), 71–91.
14. H. N. Mhaskar and E. B. Saff, Where does the L_p norm of a weighted polynomial live? *Trans. Amer. Math. Soc.* **303** (1987), 109–124.
15. P. Nevai and Geza Freud, Orthogonal polynomials and Christoffel functions: A case study, *J. Approx. Theory* **48** (1986), 3–167.
16. P. Nevai (Ed.). "Orthogonal Polynomials, Theory and Practice," NATO ASI Series, Vol. 294, Kluwer, Dordrecht, 1990.
17. P. P. Petrushev and V. Popov, "Rational Approximation of Real Functions," Cambridge Univ. Press, Cambridge, 1987.
18. E. B. Saff and V. Totik, "Logarithmic Potentials with External Fields," Springer-Verlag. [in press]
19. V. Totik, "Weighted Approximation with Varying Weight," Lecture Notes in Mathematics, Vol. 1569, Springer-Verlag, Berlin, 1994.